Numerical Realizations of the Stochastic KdV Equation With and Without Damping

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Abstract

We investigate numerical simulations of an exact solution of a stochastic Kortewg-deVries equation under Gaussian white noise. We compare the expectation values of the exact solutions to theoretical expectation values and to the numerical simulations of the stochastic Korteweg-deVries equation with and without damping. We find on average the diffused soliton vanishes long before the typically reported asymptotic limit.

$$u_t + 6uu_x + u_{xxx} = \zeta(t) - \gamma u.$$

1. Exact Solution of Stochastic KdV
2. Damped Stochastic KdV
3. Asymptotic Results - Theory
4. Numerical Solution of Damped Stochastic KdV
5. Asymptotic Results - Numerical

Exact Solution of Stochastic KdV - Wadati - 1983

$$u_t + 6uu_x + u_{xxx} = \zeta(t), \tag{1}$$

$\zeta(t)$ is Gaussian white noise: zero mean and $(<*>= E[*])$

$$< \zeta(t)\zeta(t') >= 2\epsilon\delta(t-t'). \tag{2}$$

Using the **Galilean transformation**

$$\begin{aligned} u(x,t) &= U(X,T) + W(T), \\ X &= x + m(t), \qquad T = t, \\ m(t) &= -6\int_0^t W(t')dt', \qquad W(t) = \int_0^t \zeta(t')\,dt', \end{aligned} \tag{3}$$

the stochastic KdV equation can be transformed into the KdV equation:

$$U_T + 6UU_X + U_{XXX} = 0, \tag{4}$$

$$\begin{aligned} \zeta(t) &= u_t + 6uu_x + u_{xxx} \\ &= (U+W)_T - 6WU_X + 6(U+W)U_X + U_{XXX} \\ &= U_T + 6UU_X + U_{XXX} + W_T. \end{aligned} \tag{5}$$

The One Soliton Solution Under Noise

We consider the one soliton solution of the KdV:

$$U(X,T) = 2\eta^2 \,\text{sech}^2(\eta(X - 4\eta^2 T - X_0)) \tag{6}$$

This leads directly to an exact solution of the stochastic KdV equation:

$$u(x,t) = 2\eta^2 \,\text{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right) + W(t). \tag{7}$$

Statistical Averages:

$$< u(x,t) >= 2\eta^2 \left\langle \text{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right)\right\rangle.$$

Noting

$$\text{sech}^2 z = 2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{2nz}. \tag{8}$$

Wadati then proceeded by computing.

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty}(-1)^{n+1} n \left\langle \exp\left[2n\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right]\right\rangle.$$

Statistical Averages

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} n \left\langle \exp\left[2n\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right]\right\rangle .$$

$$\begin{aligned} < W(t) > &= 0. && (9)\\ < W(t_1)W(t_2) > &= 2\epsilon \min(t_1, t_2). && (10)\\ < \exp\left(cW(t)\right) > &= \exp\left(\frac{1}{2}c^2 < W^2(t) >\right). && (11) \end{aligned}$$

This implies

$$\begin{aligned} \left\langle \exp\left(\pm 12n\eta \int_0^t W(t')\,dt'\right)\right\rangle &= \exp\left(72n^2\eta^2 \int_0^t \int_0^t < W(t_1)W(t_2) > \,dt_2 dt_1\right) \\ &= \exp(48n^2\eta^2\epsilon t^3). && (12) \end{aligned}$$

The Exact Solution for $< u(x,t) >$ via the Diffusion Equation

So far, we have

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{na+n^2 b}, \tag{13}$$

where

$$a = 2\eta(x - x_0 - 2\eta^2 t), \qquad b = 48\eta^2 \epsilon t^3.$$

Differentiating with respect to a and b leads to the initial value problem for $w(a,b) =< u(x,t) >$:

$$w_b = w_{aa}, \qquad w(a,0) = 2\eta^2 \operatorname{sech}^2 \frac{a}{2}.$$

This is solved using the Fourier transform:

$$\hat{w}(k,b) = \int_{-\infty}^{\infty} w(a,b) e^{-iak}\, da, \tag{14}$$

Thus, we have the new IVP

$$\hat{w}_b = -k^2 \hat{w}, \tag{15}$$

$$\hat{w}(k,0) = 2\eta^2 \int_{-\infty}^{\infty} \operatorname{sech}^2 \frac{a}{2} e^{-iak}\, da = 8\eta^2 \frac{\pi k}{\sinh \pi k}. \tag{16}$$

Solving the Diffusion Equation

Therefore, the transform solution is

$$\hat{w}(k,b) = 8\eta^2 \frac{\pi k}{\sinh \pi k} e^{-bk^2} \tag{17}$$

and thus we have

$$< u(x,t) >= \frac{4\eta^2}{\pi} \int_{-\infty}^{\infty} \frac{\pi k}{\sinh \pi k} e^{iak-bk^2}\, dk. \tag{18}$$

Using the transform pairs

$$f(a) = 2\eta^2 \operatorname{sech}^2 \frac{a}{2} \quad \Leftrightarrow \quad \hat{f}(k) = 8\eta^2 \frac{\pi k}{\sinh \pi k}$$

$$g(a,b) = \frac{1}{\sqrt{4\pi b}} e^{-a^2/4b} \quad \Leftrightarrow \quad \hat{g}(k,b) = e^{-bk^2}.$$

The Convolution Theorem gives the exact result

$$\begin{aligned} < u(x,t) > &= (f*g)(a) \\ &= \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{-(a-s)^2/4b} \operatorname{sech}^2 \frac{s}{2}\, ds, \end{aligned} \tag{19}$$

where

$$a = 2\eta(x - x_0 - 2\eta^2 t), \qquad b = 48\eta^2 \epsilon t^3.$$

The Damped Stochastic KdV

$$u_t + 6uu_x + u_{xxx} = \zeta(t) - \gamma u, \tag{20}$$

can be transformed into the damped KdV equation:

$$U_T + 6UU_X + U_{XXX} = -\gamma U, \tag{21}$$

using the Galilean transformation

$$\begin{aligned} u(x,t) &= U(X,T) + W(T), \\ X = x + m(t), \qquad W(t) &= e^{-\gamma t}\int_0^t \zeta(t')e^{\gamma t'}\,dt', \qquad m(t) = -6\int_0^t W(t')dt'. \end{aligned} \tag{22}$$

The leading order solution

$$U_0(X,T) = 2\eta^2(\tau)\,\text{sech}^2\left[\eta(\tau)X + X_0(\tau)\right], \qquad \eta_\tau = -\frac{2}{3}\gamma\eta, \qquad X_{0\tau} = -4\eta^2 + \frac{\gamma}{3\eta}, \tag{23}$$

leads to the exact average

$$< u(x,t) >= \frac{\eta^2}{\sqrt{\pi b}}\int_{-\infty}^{\infty} e^{-(a-s)^2/4b}\,\text{sech}^2\frac{s}{2}\,ds \tag{24}$$

where $a = 2\eta(x - x_0 - 2\eta^2 t), \qquad b = \frac{72\epsilon\eta^2}{\gamma^3}\left[2\gamma t - 3 + 4e^{-\gamma t} - e^{-2\gamma t}\right]$

Asymptotics

Large Time Behavior of the Soliton Peak:

$$< u(x,t) >_{\max} = < u(x,t) > |_{a=0} = \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{-s^2/4b} \operatorname{sech}^2 \frac{s}{2}\, ds \tag{25}$$

where

$$b = \begin{cases} 48\eta^2 \epsilon t^3, & \text{No Damping} \\ \frac{72\epsilon\eta^2}{\epsilon\gamma^3}\left[2\gamma t - 3 + 4e^{-\gamma t} - e^{-2\gamma t}\right], & \text{Damping} \end{cases}$$

For large times ($b = 48\eta^2\epsilon t^2 \gg 1$ or $b \sim 144\epsilon\eta^2 t/\gamma^2 \gg 1$):

$$< u(x,t) > = \frac{4\eta^2}{\sqrt{\pi}} \left(1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2)B_n \pi^{2n}}{(2n)!} \frac{\partial^n}{\partial b^n}\right) \frac{e^{-a^2/4b}}{\sqrt{b}}. \tag{26}$$

Most focus on the $t \longrightarrow \infty$ result that

$$< u(x,t)_{\max} > \sim \begin{cases} \frac{\eta}{\sqrt{3\pi\epsilon}} t^{-3/2}, & \text{No Damping} \\ \frac{\eta_0\gamma}{12\sqrt{\pi\epsilon}} t^{-1/2} e^{-2\gamma t/3}, & \text{Damping} \end{cases}$$

Numerical Simulation of the Stochastic KdV

Zabusky and Kruskal (1965) studied the KdV equation using the finite difference approximation [2]:

$$u_t = \frac{u(j,n+1) - u(j,n-1)}{2\Delta t} + O(\Delta t^2) \tag{27}$$

$$u = \frac{u(j+1,n) + u(j,n) + u(j-1,n)}{3} + O(\Delta x^2) \tag{28}$$

$$u_x = \frac{u(j+1,n) - u(j-1,n)}{2\Delta x} + O(\Delta x^2) \tag{29}$$

$$u_{xxx} = \frac{u(j+2,n) - 2u(j+1,n) + 2u(j-1,n) - u(j-2,n)}{2\Delta x^3} + O(\Delta x^2), \tag{30}$$

where $t = n\Delta t$ and $x = a + j\Delta x$.

Stability Condition: $\Delta t = \frac{(\Delta x)^3}{4}$

Herman and Kickerbocker (1990) noted the velocity shift

$$v = \frac{dx_c}{dt} = 4\eta^2 - \frac{4}{5}\eta^4 \Delta x^2, \tag{31}$$

Simulating Brownian Motion

We use discretized Brownian motion, where $W(t)$ is specified at discrete t values.

Let, $\delta t = T/N$ for some positive integer N and let W_j denote $W(t_j)$ with $t_j = j\delta t$.

Conditions for Brownian motion tell us that

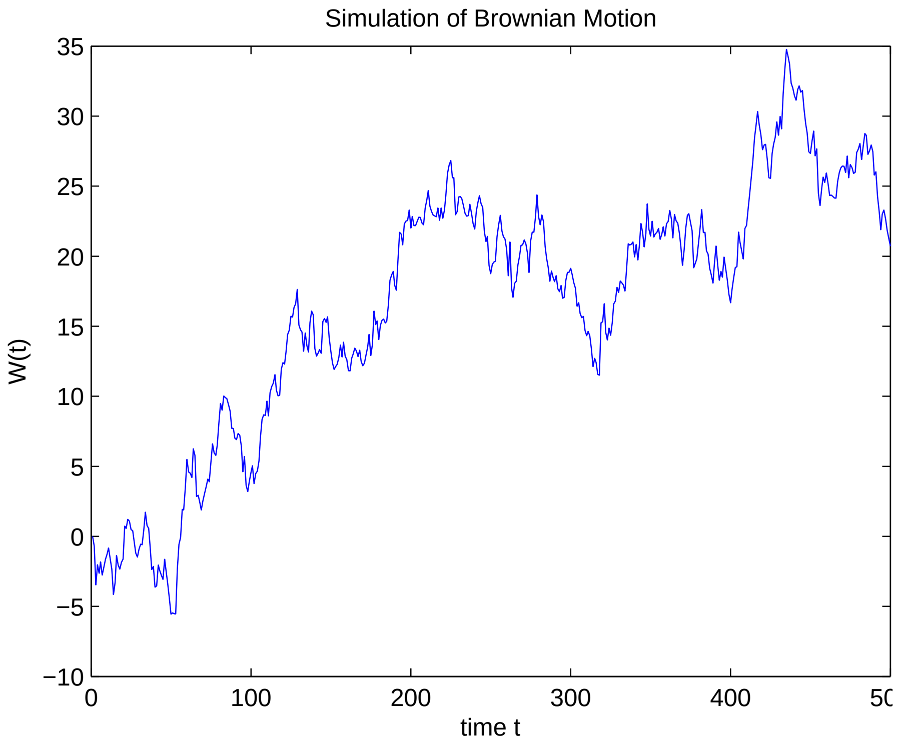
1. $W_0 = 0$ with probability 1, and
2. $W_j = W_{j-1} + dW_j,\ j = 1, 2, \ldots, N$

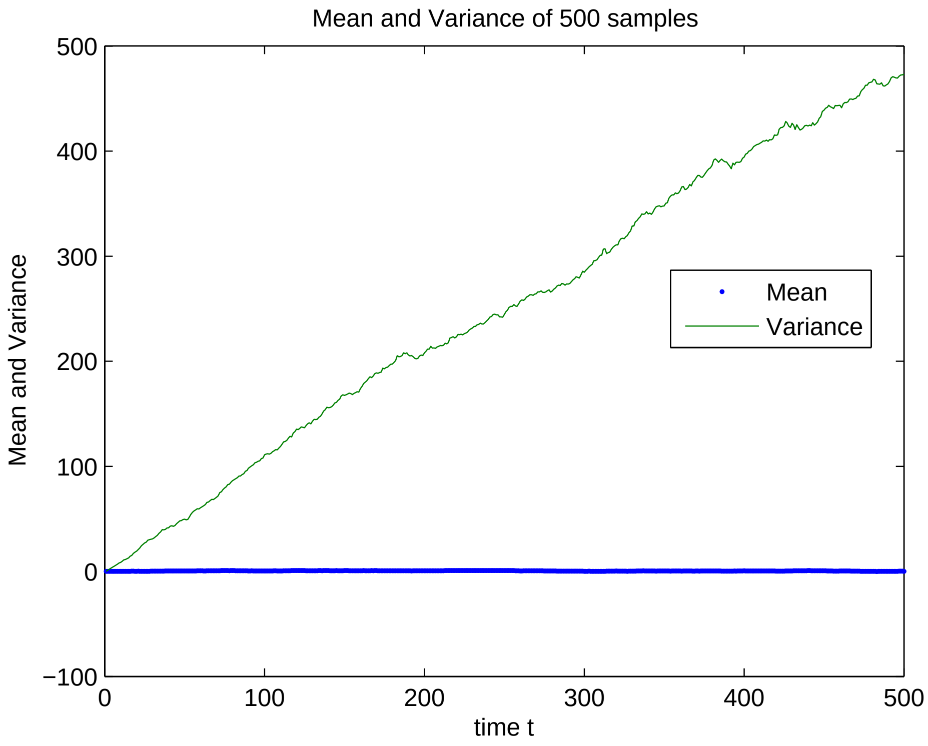
where each dW_j is an independent random variable of the form $\sqrt{\delta t}N(0,1)$, where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

$$
\begin{aligned}
&W_{t+1} - W_t = \Delta W_t, \\
&E\{\Delta W_t\} = 0, \\
&E\{\Delta W_t^2\} = \Delta t \\
&W_{t+1} = W_t + \sqrt{\Delta t}.N(0,1), \\
&W_0 = 0
\end{aligned}
$$

Brownian Motion - Simulation, Mean, and Variance

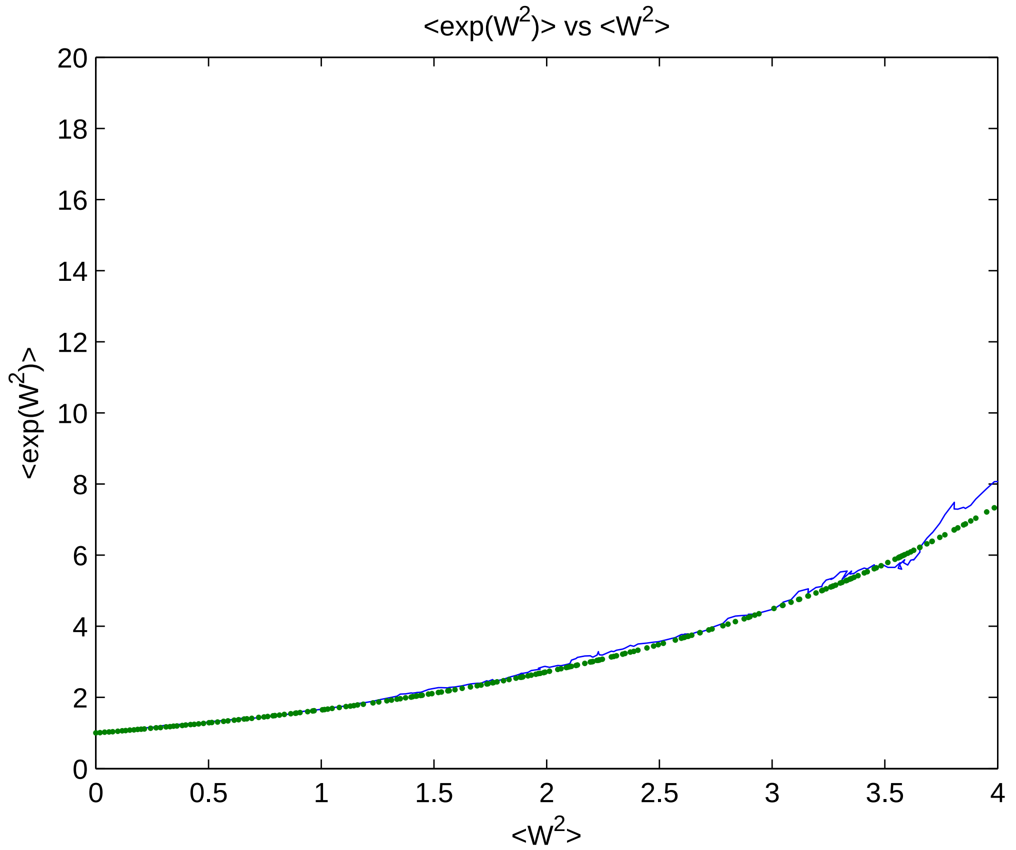
```
randn('state',100)              % set the state of randn
T = 1; N = 500; dt = T/N;
dW = sqrt(dt)*randn(1,N);       % increments
W = cumsum(dW);                 % cumulative sum
```





Wadati Identity

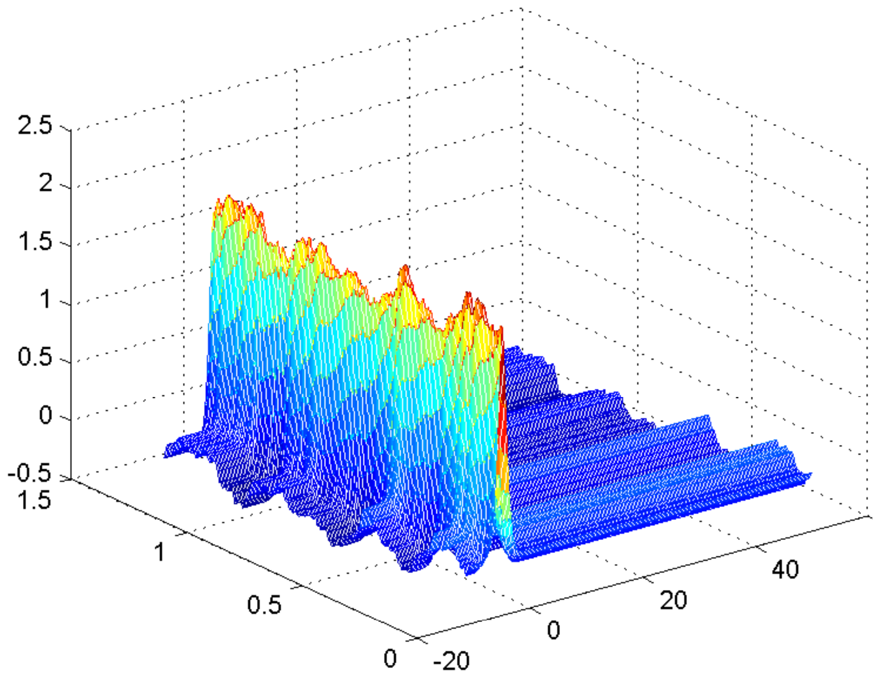
The other identity to confirm is $< \exp(cW(t)) >= \exp(\frac{1}{2} < W^2(t) >)$.

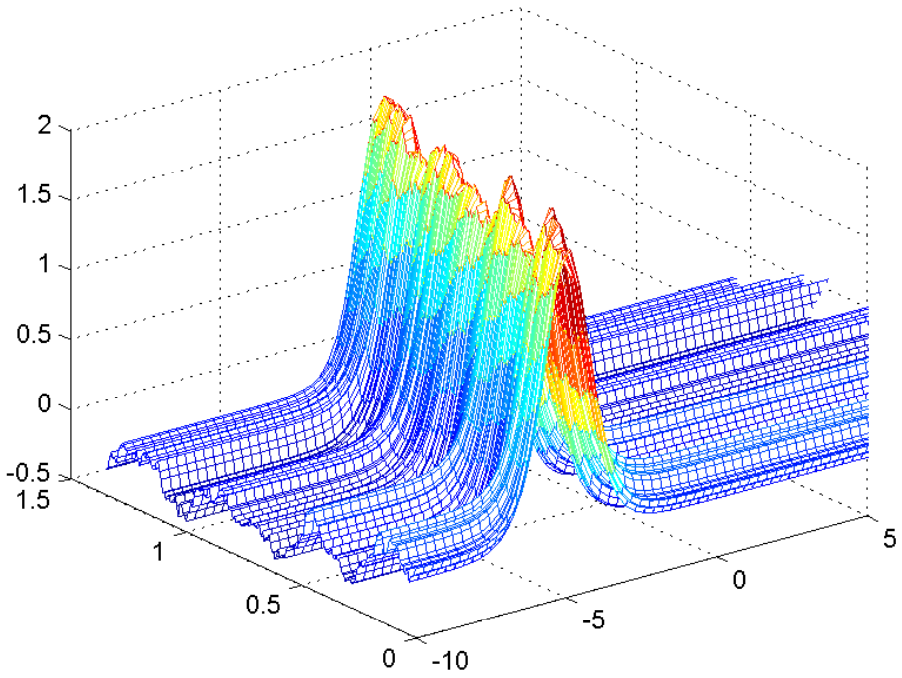


Numerical Results - Code for Averaged Soliton

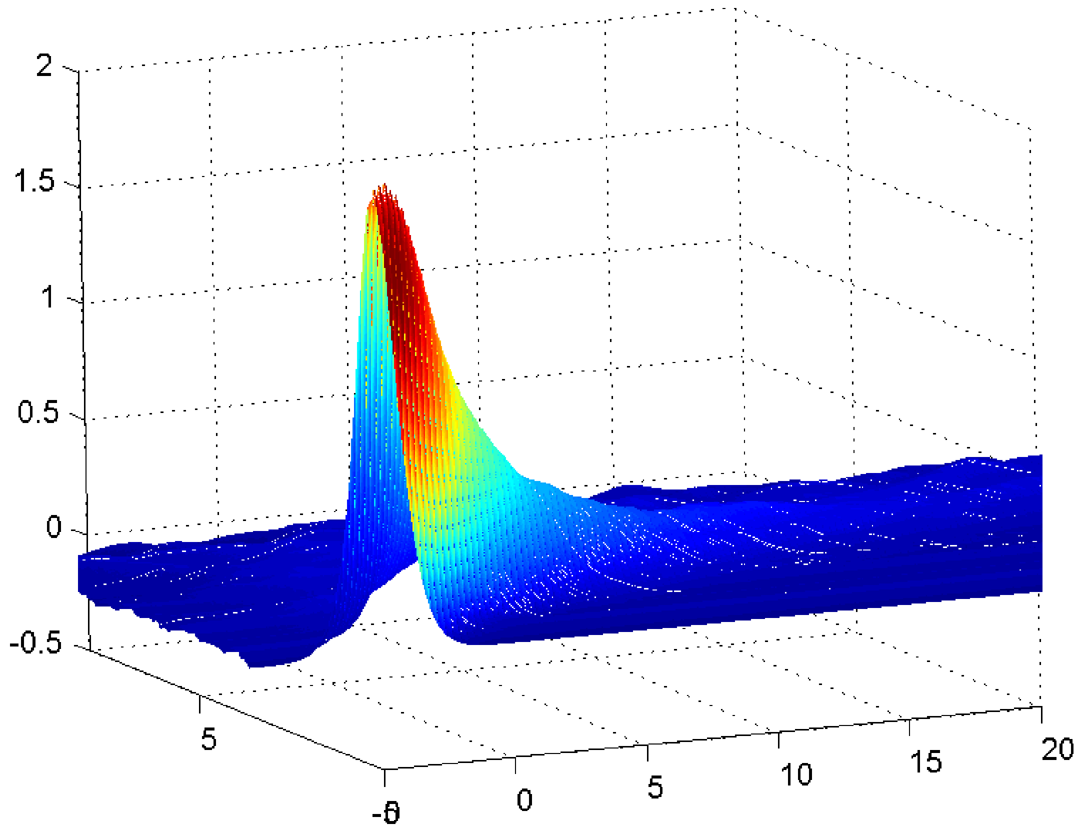
```
mu=sqrt(2*eps);

uu=zeros(Tsteps,N+1);
for k=1:M
   r=randn(Tsteps,1);
   w=zeros(Tsteps,1);
   Iw=zeros(Tsteps,1);
   w(1)=0;
   Iw(1)=w(1)/2*dt;
   for i=2:Tsteps
       w(i)=w(i-1)+mu*sqrt(dt)*r(i);
       Iw(i)=Iw(i-1)+(w(i-1)+w(i))/2*dt;
   end

   u=zeros(Tsteps,N+1);
   for i=1:Tsteps;
      for j=1:N+1;
      u(i,j)=w(i)+2*zeta^2*(sech(zeta*(x(j)-4*zeta^2*t(i)-x0-
   6*Iw(i)))).^2;
      end
   end
   uu=uu+u;
end
uu=uu/M;
```

Single Soliton Plus Noise

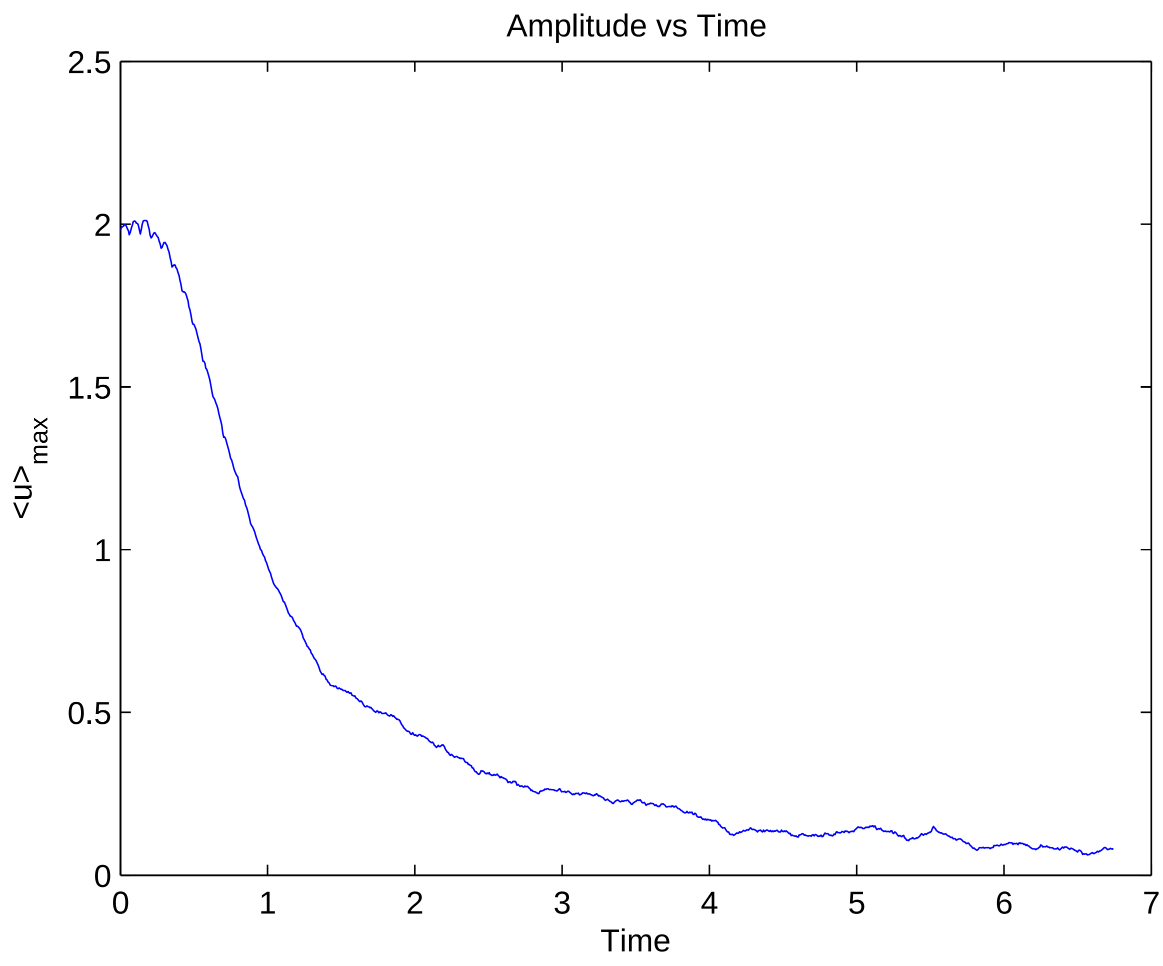




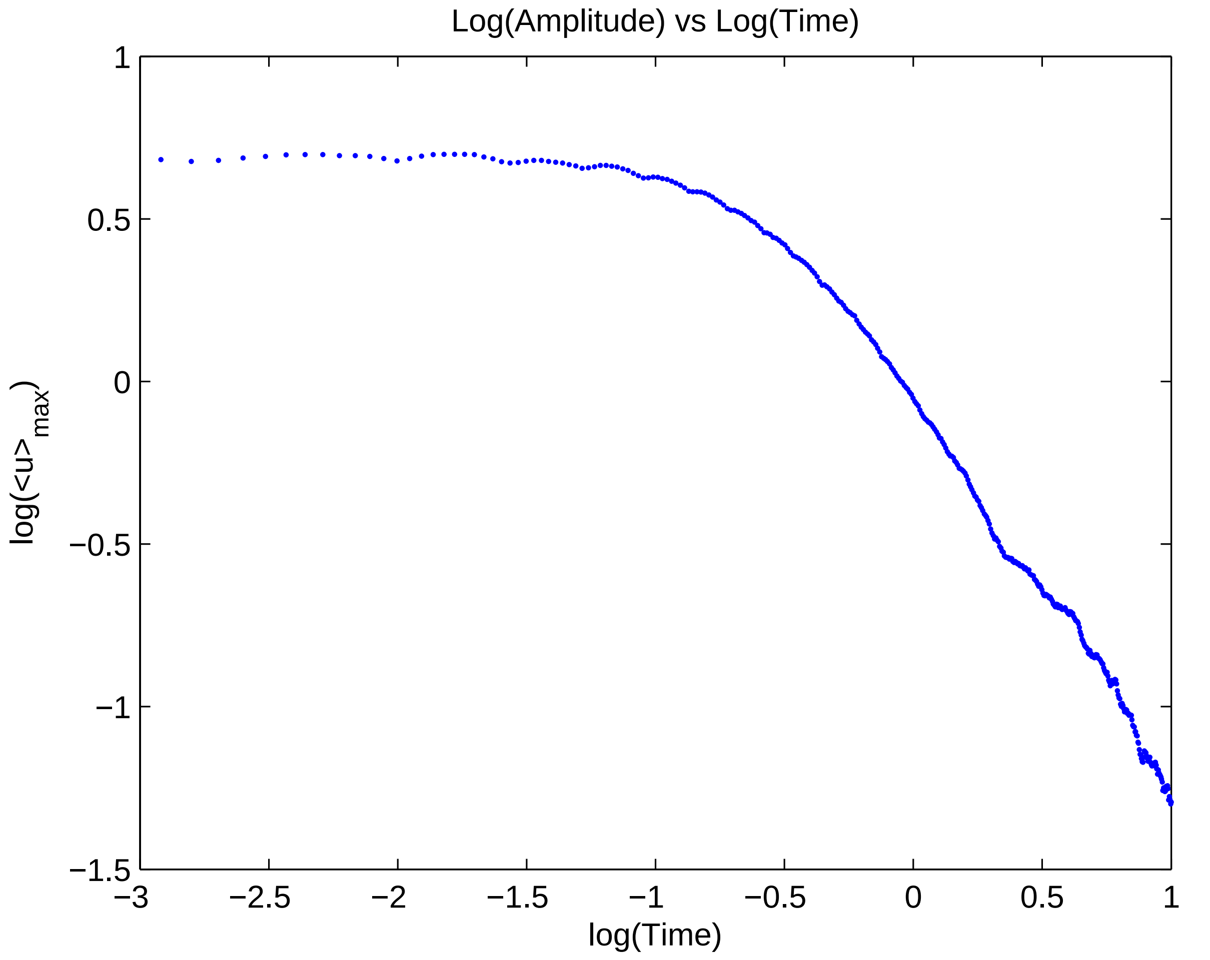
Averaged Soliton with Noise



Average Soliton Amplitude



Loglog Plot for Average Soliton Amplitude



Zabusky-Kruskal Scheme Plus Stochastic Terms

```
for count=1:runs
    % Generate Gaussian noise
    for j=1:Tsteps
        dW(j)=mu*sqrt(dt)*randn;
    end;
    % Time Loop
    for i=3:Tsteps
      % Generate solution using Zabusky-Kruskal scheme
      for j=3:N-1
       u2(j)=u0(j)-((2*dt*u1(j+1))/(dx))*(u1(j+1)-u1(j-1)+...
       u1(j)+u1(j-1)-1/(dx^2))+((2*dt*u1(j-1))/(dx))*...
       (u1(j)+u1(j-1)-1/(dx^2))-(dt*u1(j+2))/dx^3+...
       (dt*u1(j-2))/dx^3+(dW(i)+dW(i-1));
    end;
end;
```

Vectorized Form

```
A=(spdiags(-2*ones(N1),1,N1,N1)+spdiags(ones(N1),2,N1,N1)
  ...+spdiags(2*ones(N1),-1,N1,N1)+spdiags(-ones(N1),-2,N1,N1))*dt/dx^3;
B=(spdiags(ones(N1),1,N1,N1)+spdiags(-ones(N1),-1,N1,N1))*dt/dx;
C=(spdiags(ones(N1),1,N1,N1)+spdiags(ones(N1),0,N1,N1)
  ...+spdiags(ones(N1),-1,N1,N1))*2;
D=spdiags(ones(Tsteps),0,Tsteps,Tsteps)+spdiags(ones(Tsteps),-1,Tsteps,Tsteps);

for chunk=1:NChunks
  for i=istart:Tsteps
    for count=1:runs
    dW=mu*sqrt(dt)*randn(1,Tsteps);
    % Zabusky-Kruskal scheme:
    %     terms are given in vectorized form as
    %   UXXX=A*u1;
    %   UX=B*u1;
    %   U=C*u1;
    %   u2 = uu0-UXXX-U.*UX;
    u2 = u0 -A*u1-(C*u1).*(B*u1)+ddW(i)*ones(size(u0));
    end;
  end;
end;
```

Stochastic KdV Results

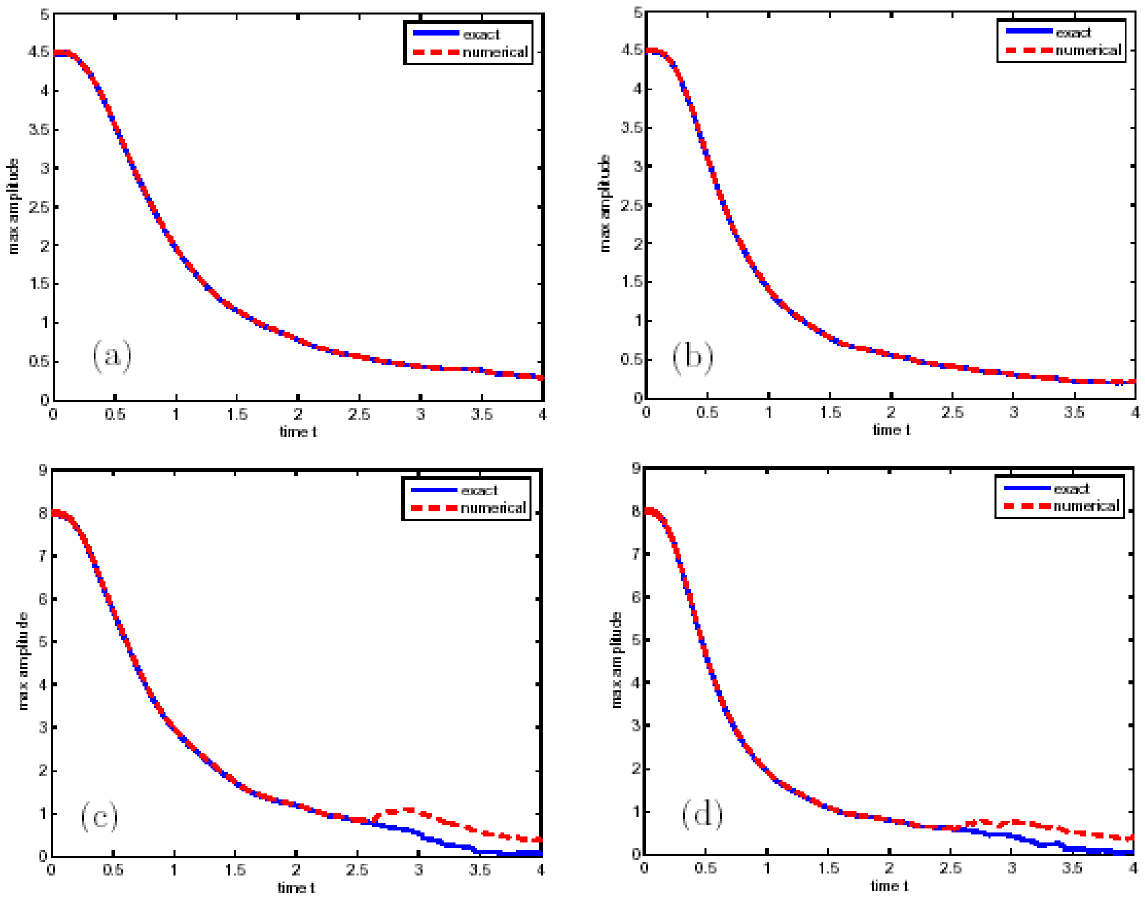


Figure 1: Plot of $< u(x,t) >_{max}$ vs. t for both the exact and numerical solutions with $x \in [-10, 40]$ and $N = 500$: (a): $\eta = 1.5$, $\epsilon = 0.05$, (b): $\eta = 1.5$, $\epsilon = 0.1$, (c): $\eta = 2$, $\epsilon = 0.05$, (d): $\eta = 2$, $\epsilon = 0.1$.

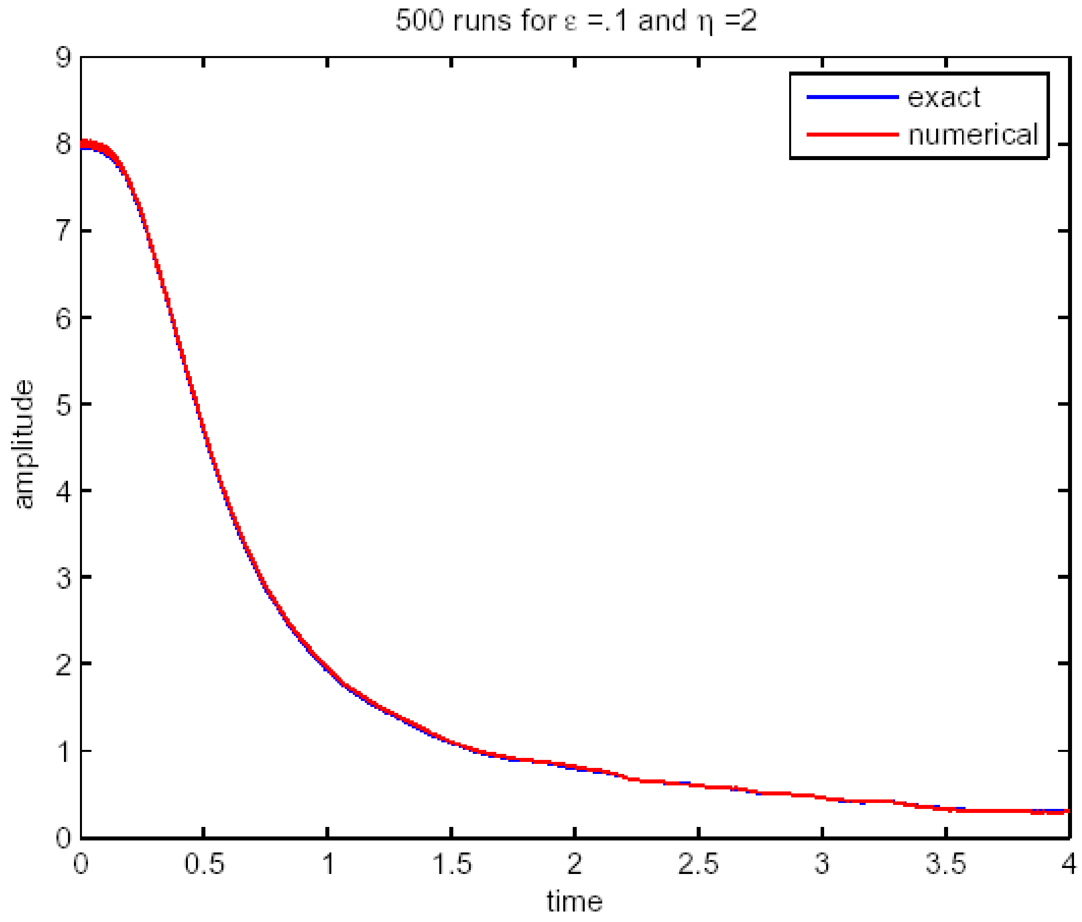


Figure 2: Amplitude vs. time plot for 500 runs for $x \in [-10, 90]$ with $N = 1000$, $\epsilon = .1$, and $\eta = 2$.

Damped, Stochastic KdV Amplitudes - Varying N

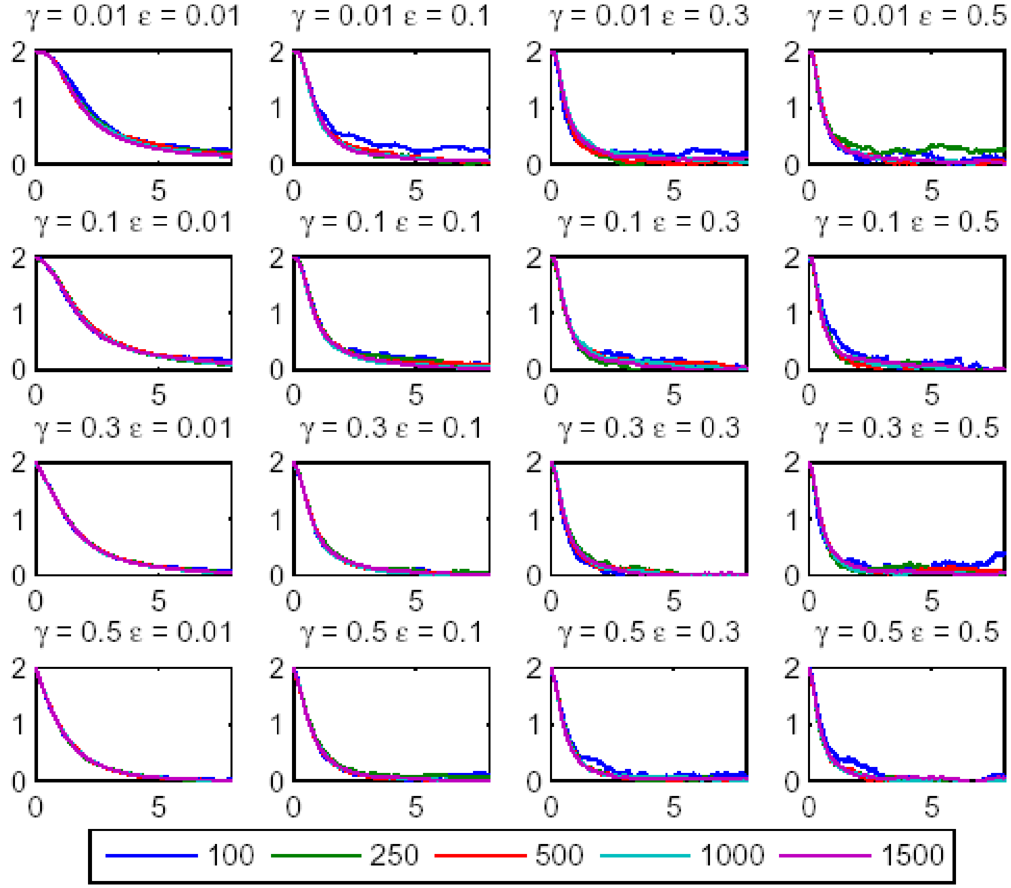


Figure 3: Amplitude vs time plots for varying ϵ and γ with $N = 100, 200, 500, 1000, 1500$.

Comparisons of Amplitude Decay Due to Noise and Damping

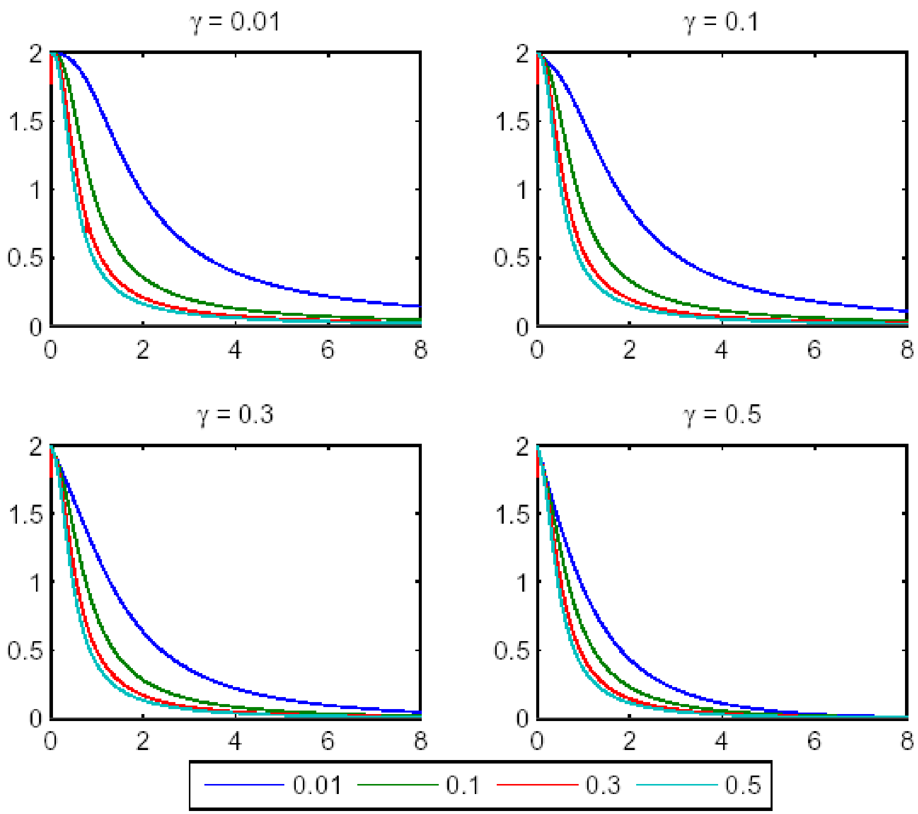


Figure 4: $\epsilon = 0.01, 0.1, 0.3, 0.5$

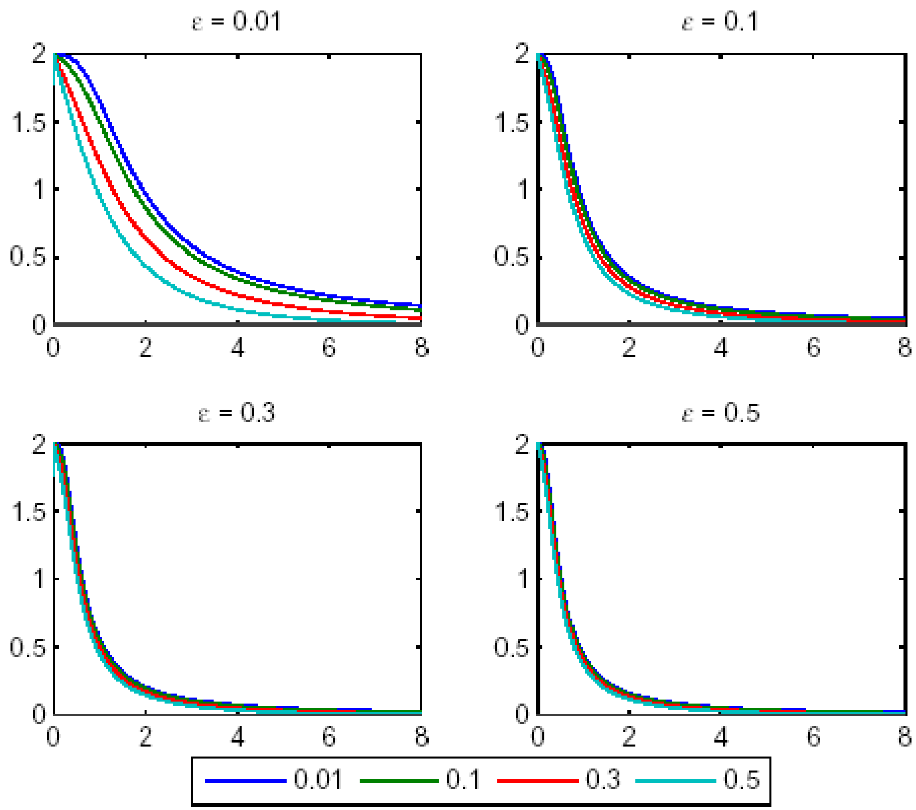
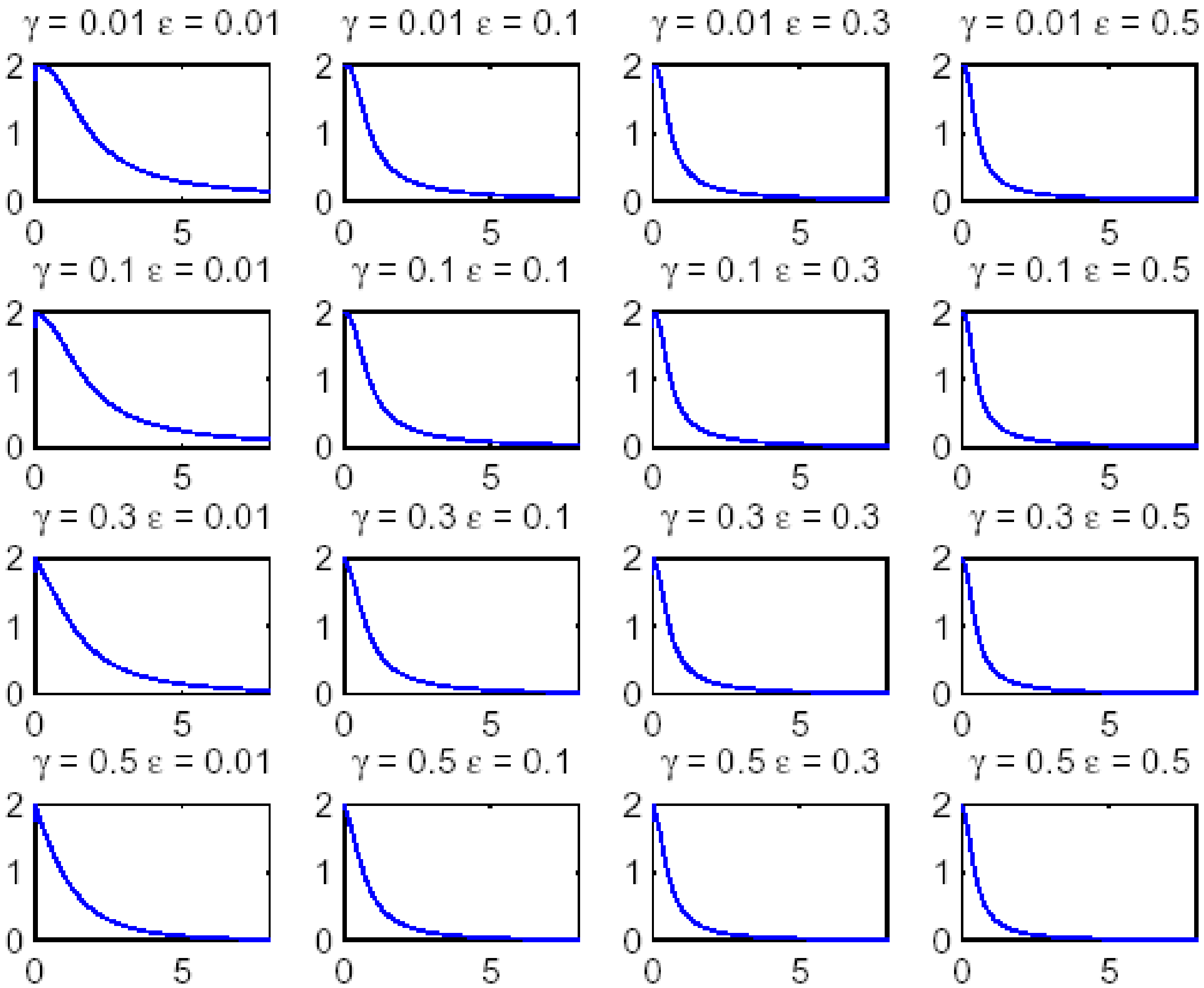
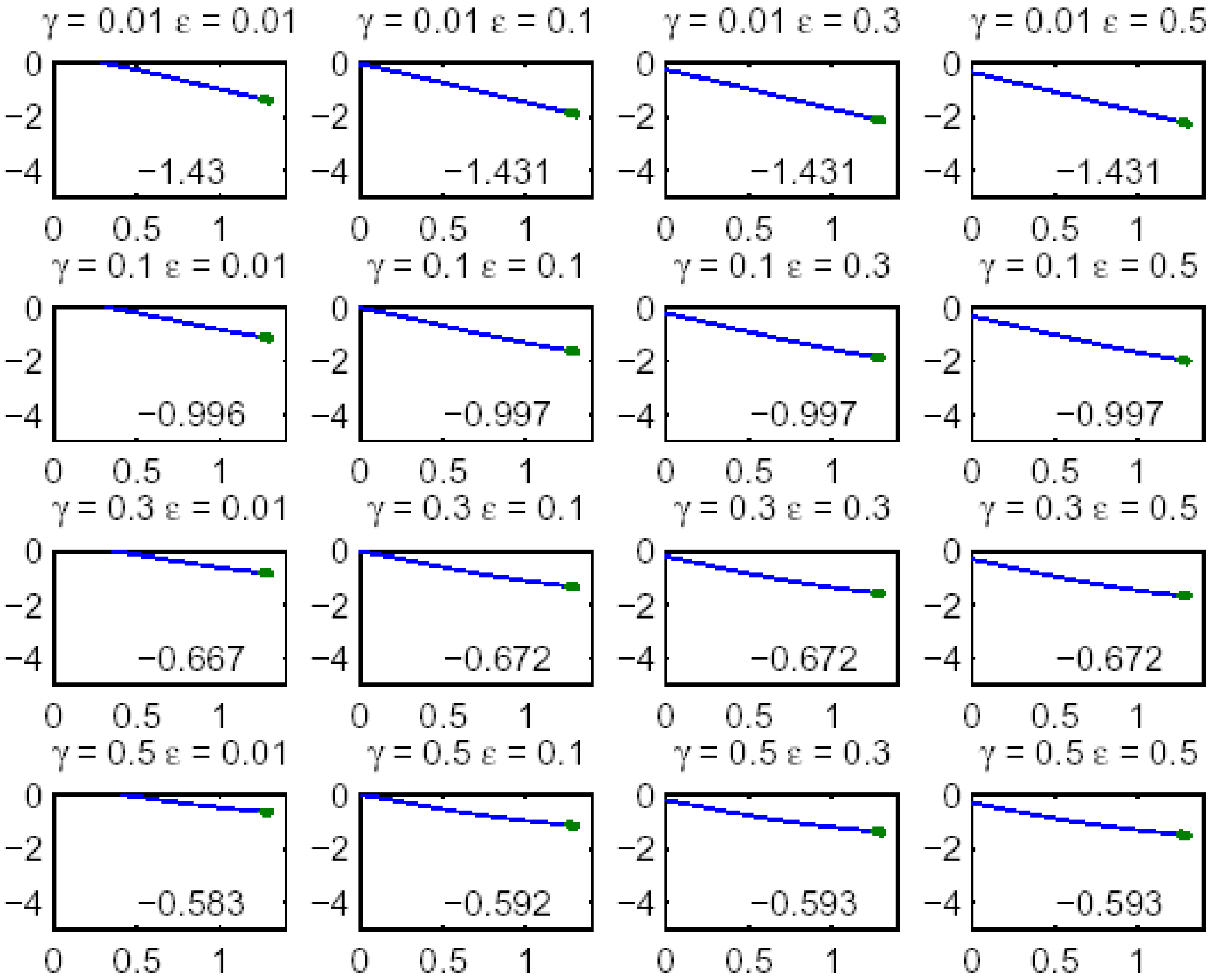


Figure 5: $\gamma = 0.01, 0.1, 0.3, 0.5$

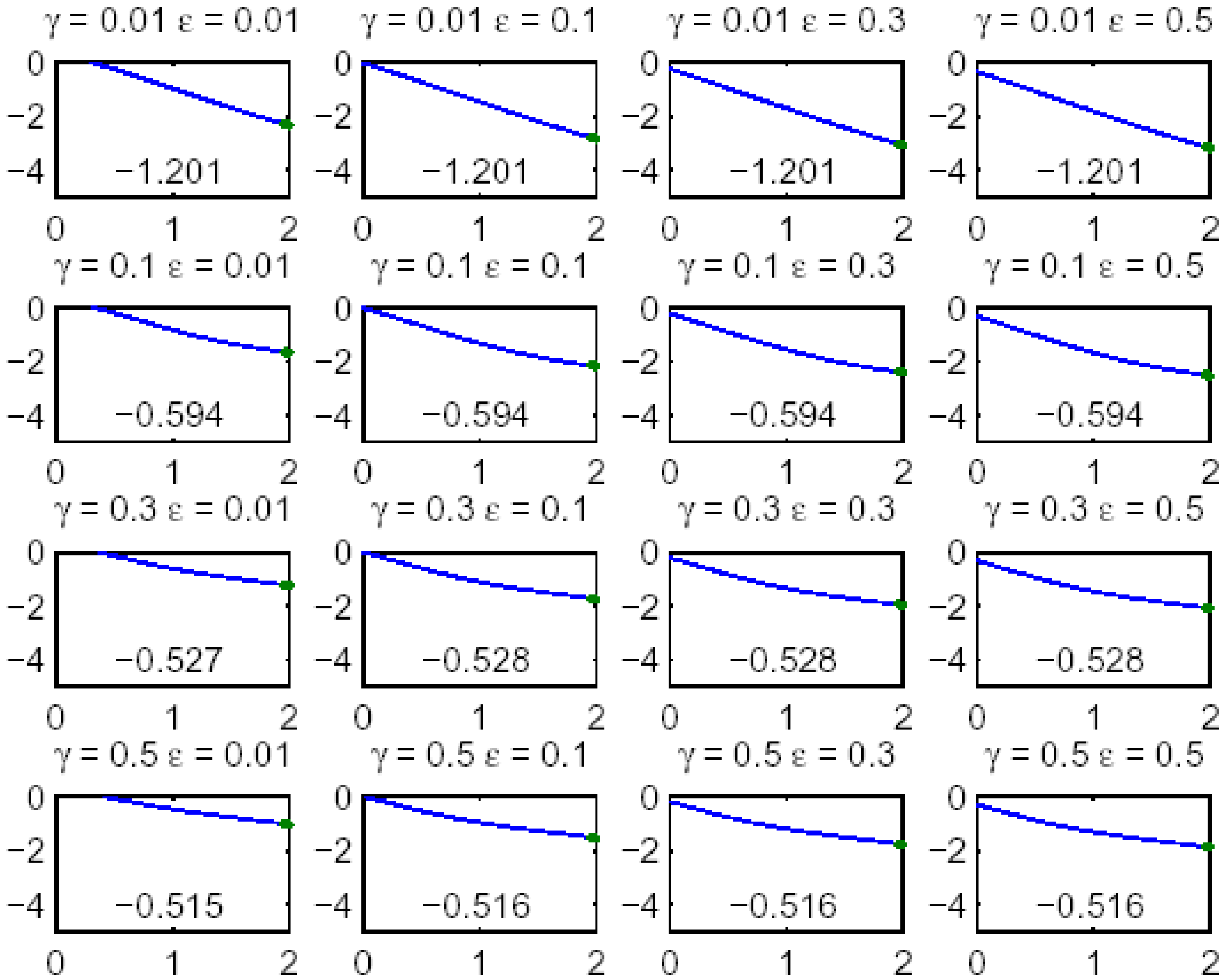
Amplitude Decay for Several Parameters - Integrated



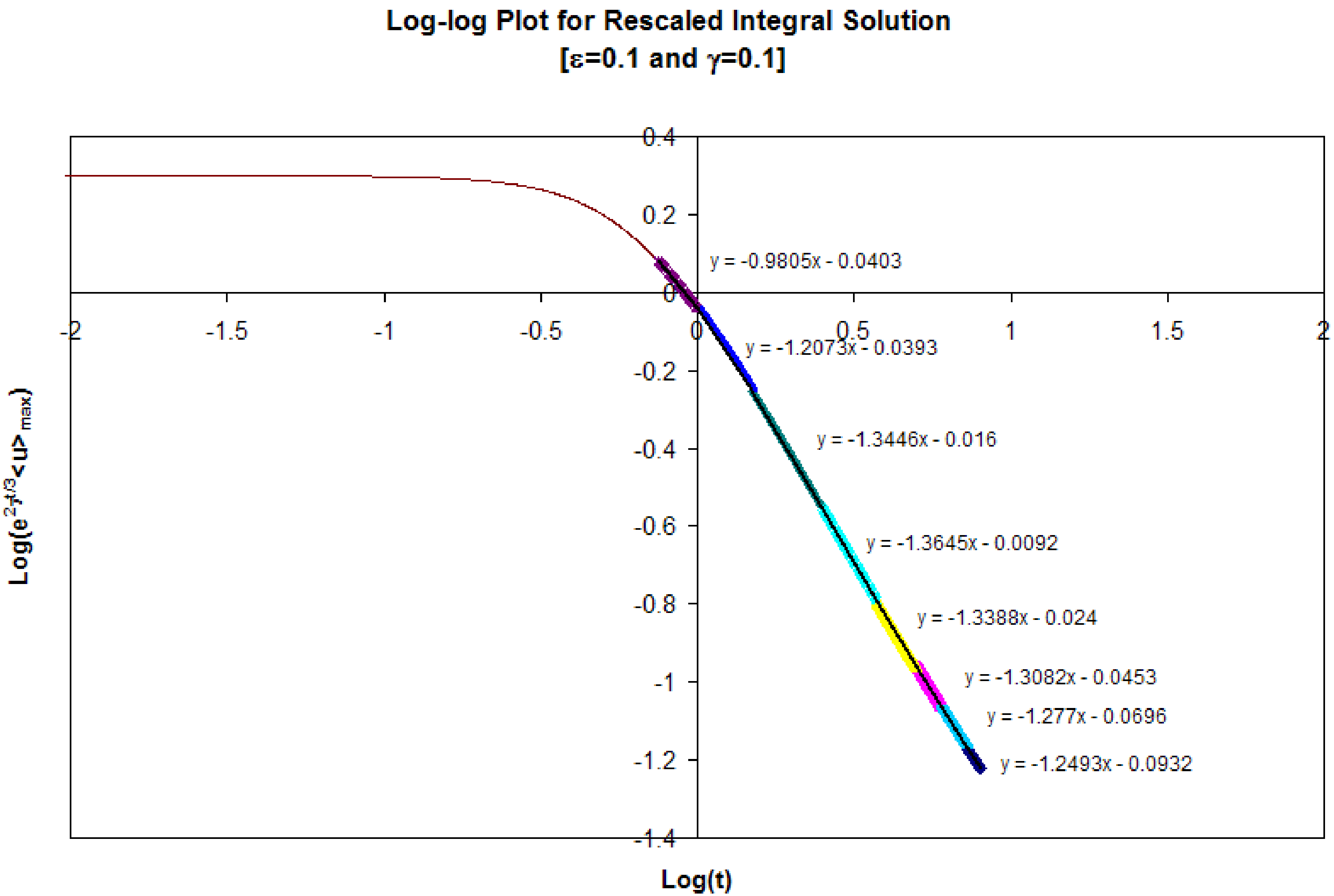
Decay Constants for $t \in [0, 10]$



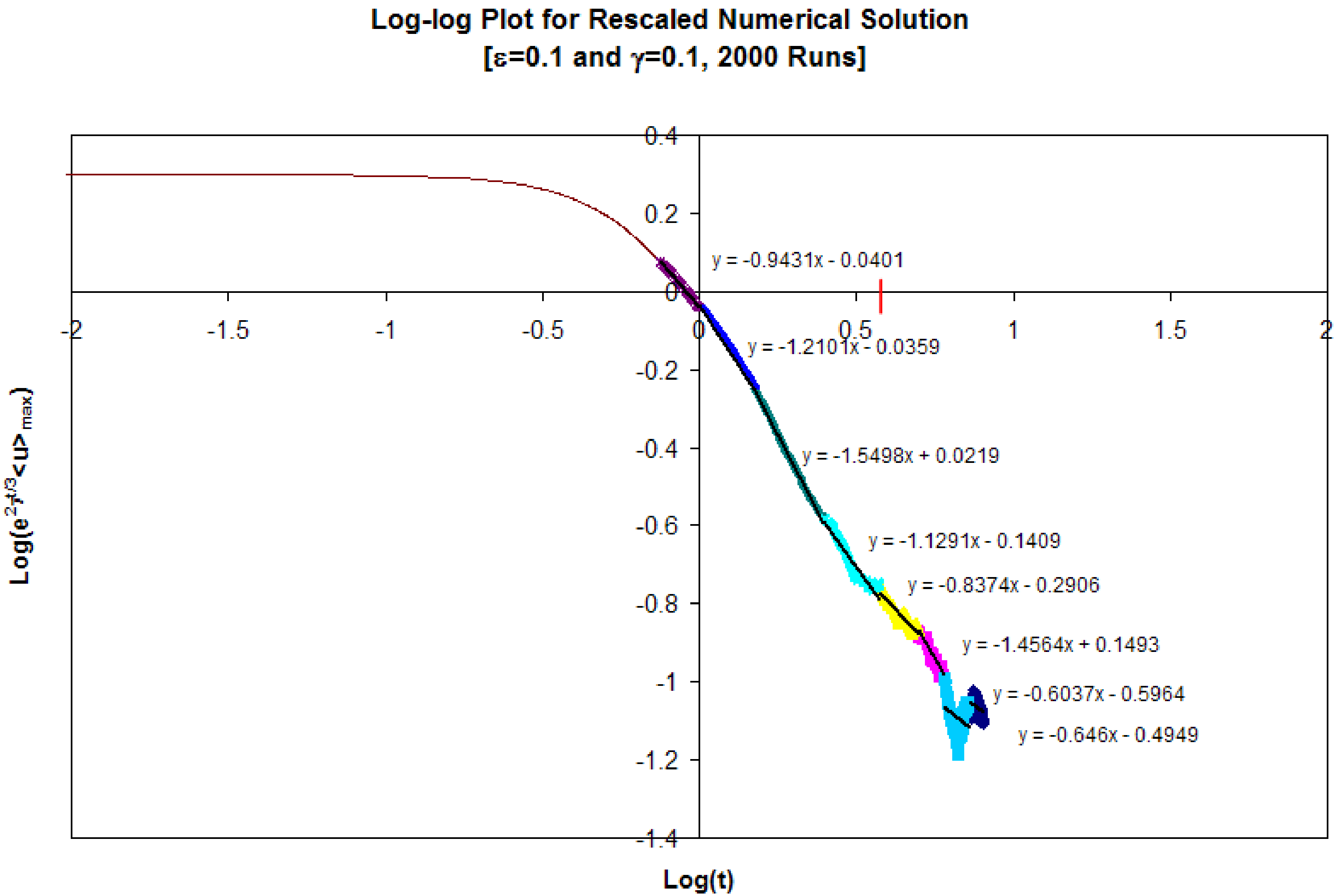
Decay Constants for $t \in [0, 100]$



Decay Constants for Integral Computation



Decay Constants for Finite Difference Results



The Two Soliton Solution of the KdV Equation

When two solitons collide, they interact elastically. The exact solution for the two soliton equation is given by

$$u(x,t) = \frac{2(p^2-q^2)(p^2+q^2 \,\text{sech}^2\chi(x,t)\sinh^2\theta(x,t))}{(p\cosh\theta(x,t) - q\tanh\chi(x,t)\sinh\theta(x,t))^2} \tag{32}$$

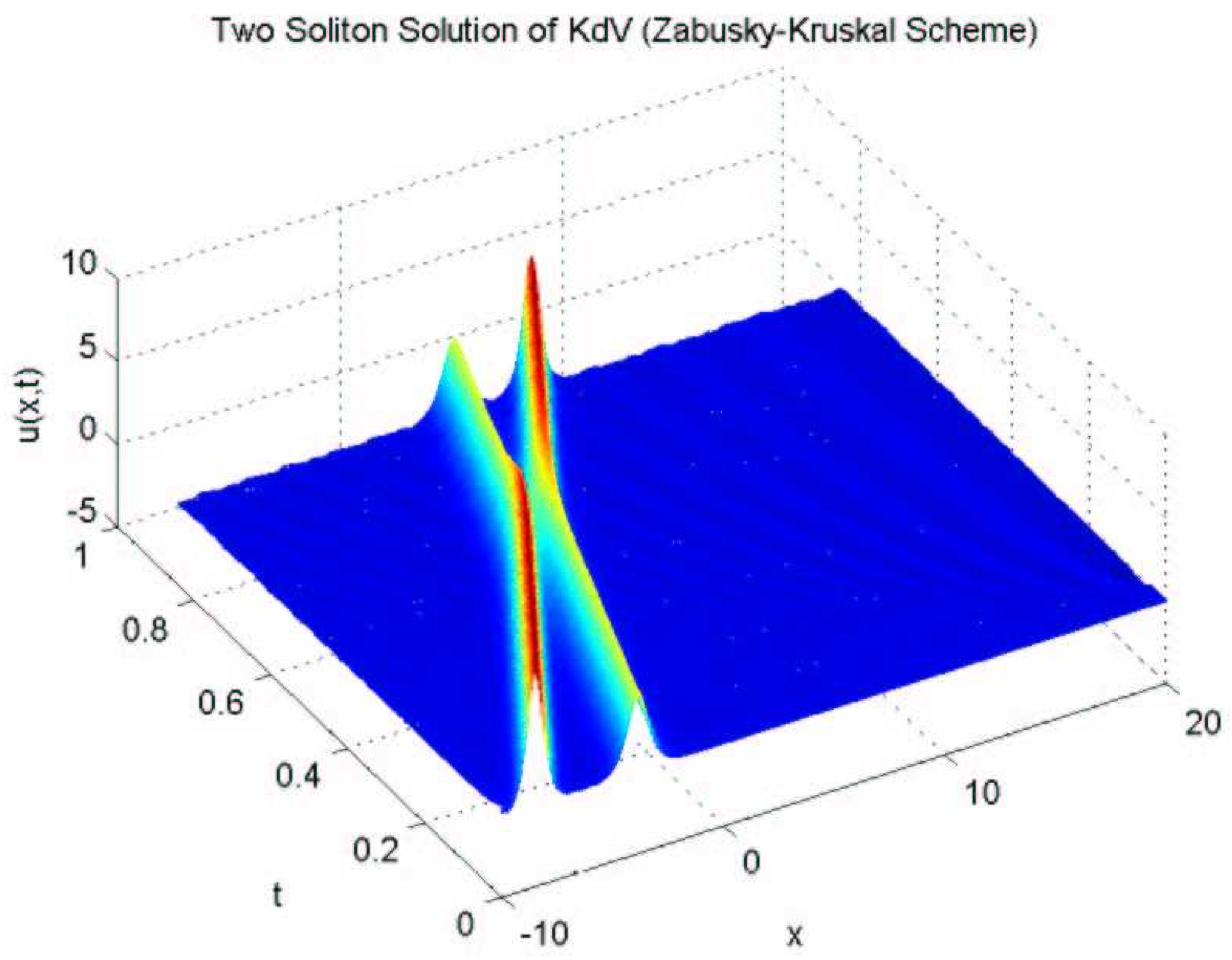
where the phases are

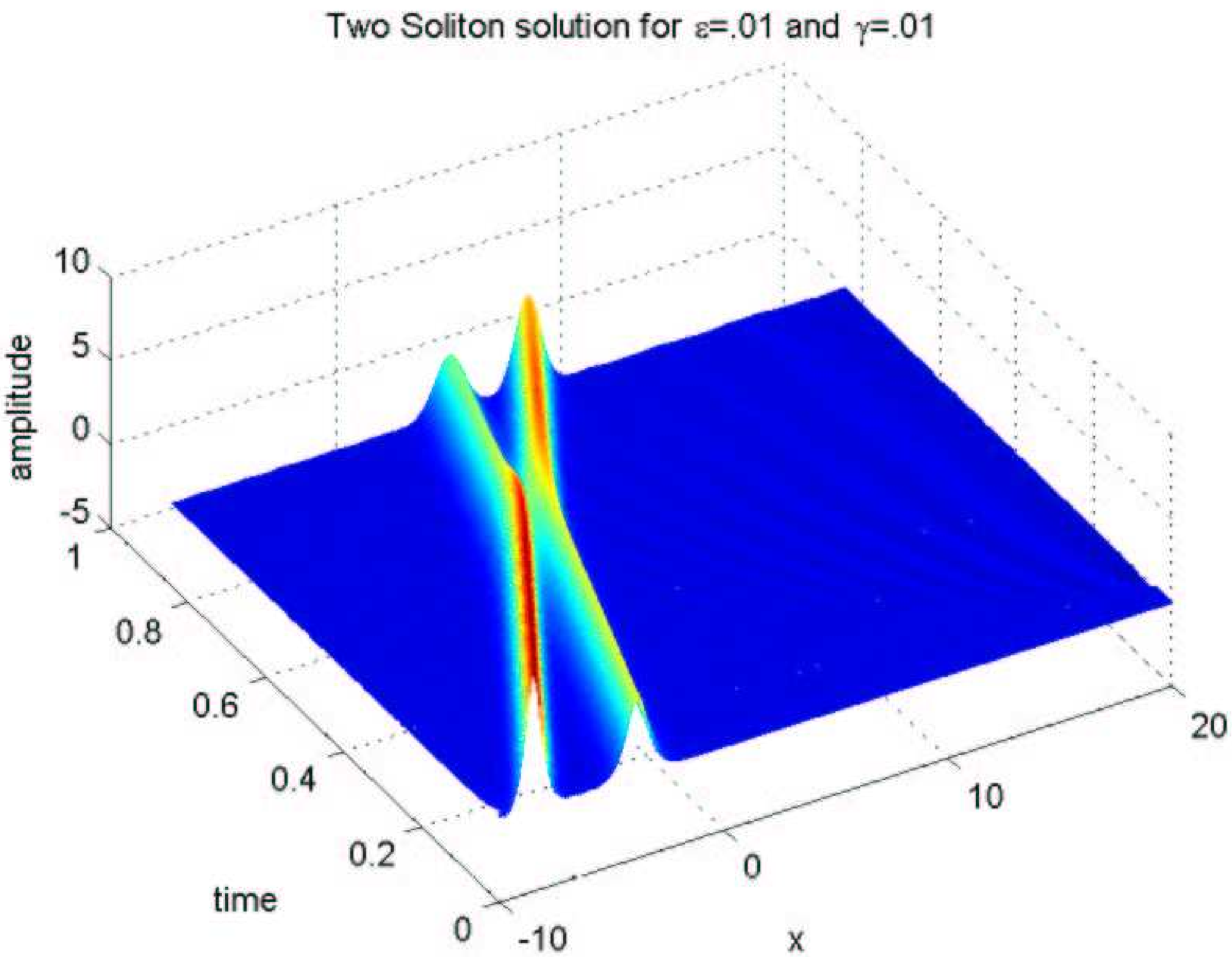
$$\theta(x,t) = px - 4p^3(t-t_0) \tag{33}$$

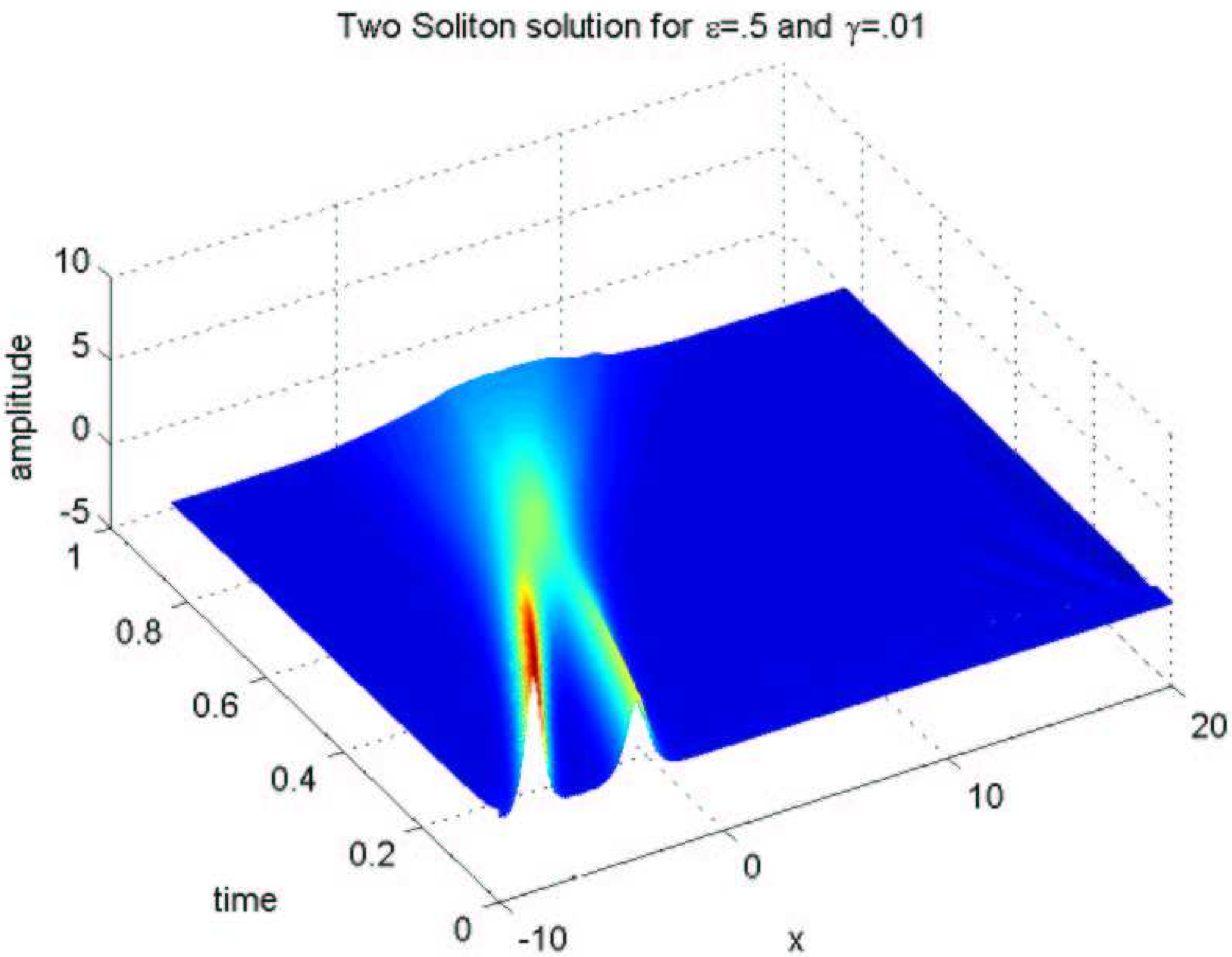
and

$$\chi(x,t) = qx - 4q^3(t-t_0). \tag{34}$$

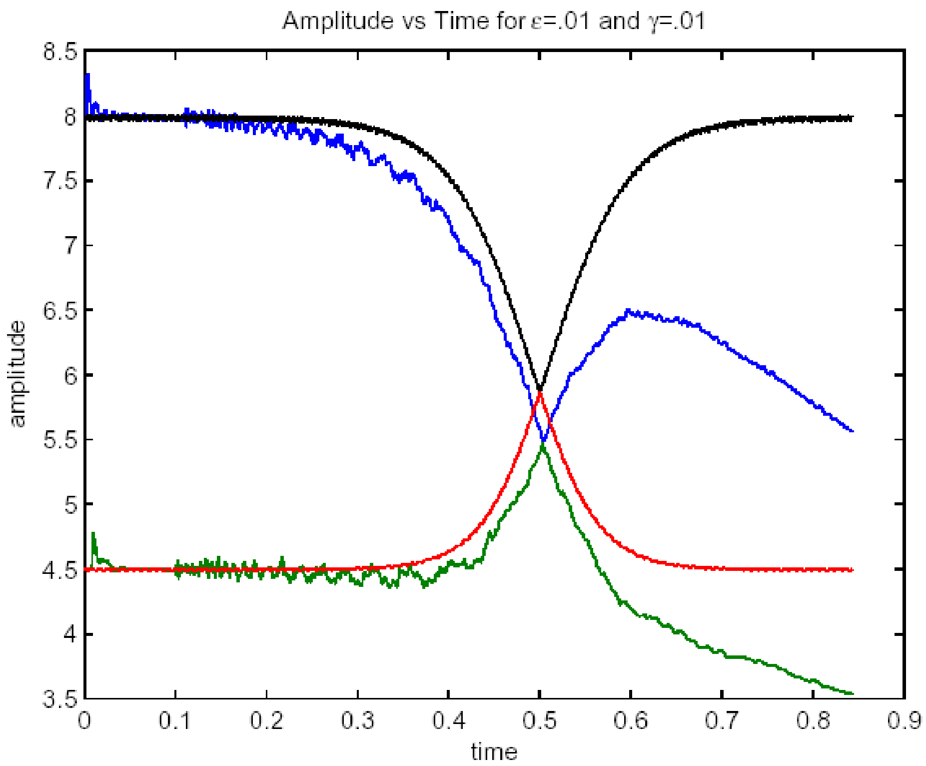
In our simulations we take $p = 2$, $q = 1.5$ and $t_0 = 0.5$.

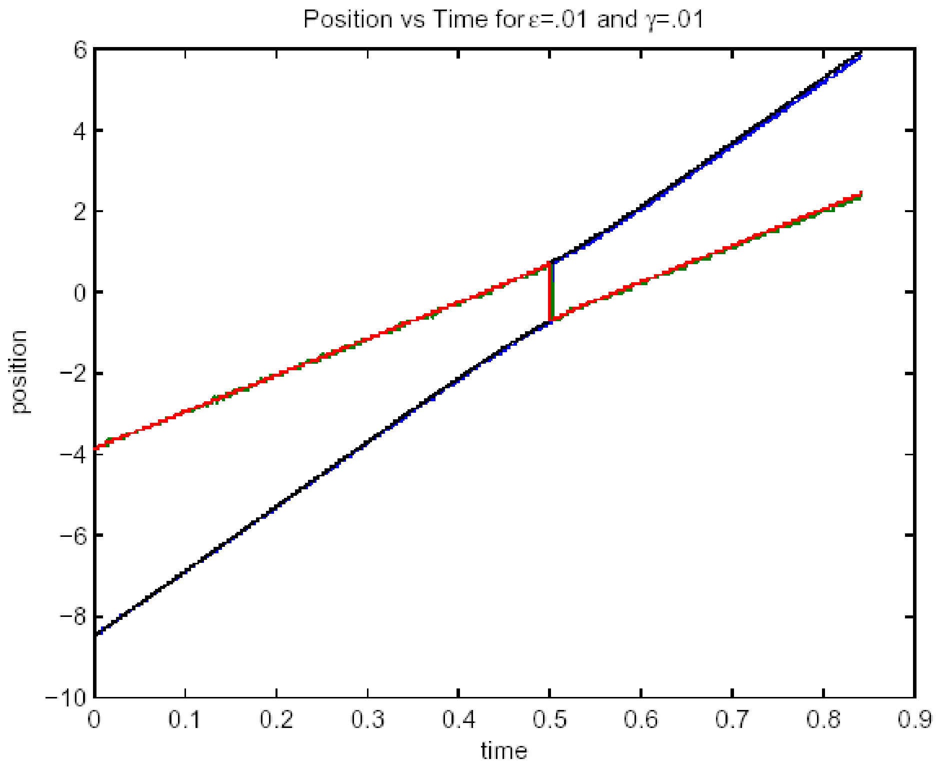

Two Soliton Solution of KdV (Zabusky-Kruskal Scheme)
u(x,t)
10
5
0
-5
1
0.8
0.6
0.4
0.2
t
0
-10
0
10
20
x


Two Soliton solution for ε=.01 and γ=.01
amplitude
10
5
0
-5
1
0.8
0.6
0.4
0.2
0
time
-10
0
10
20
x

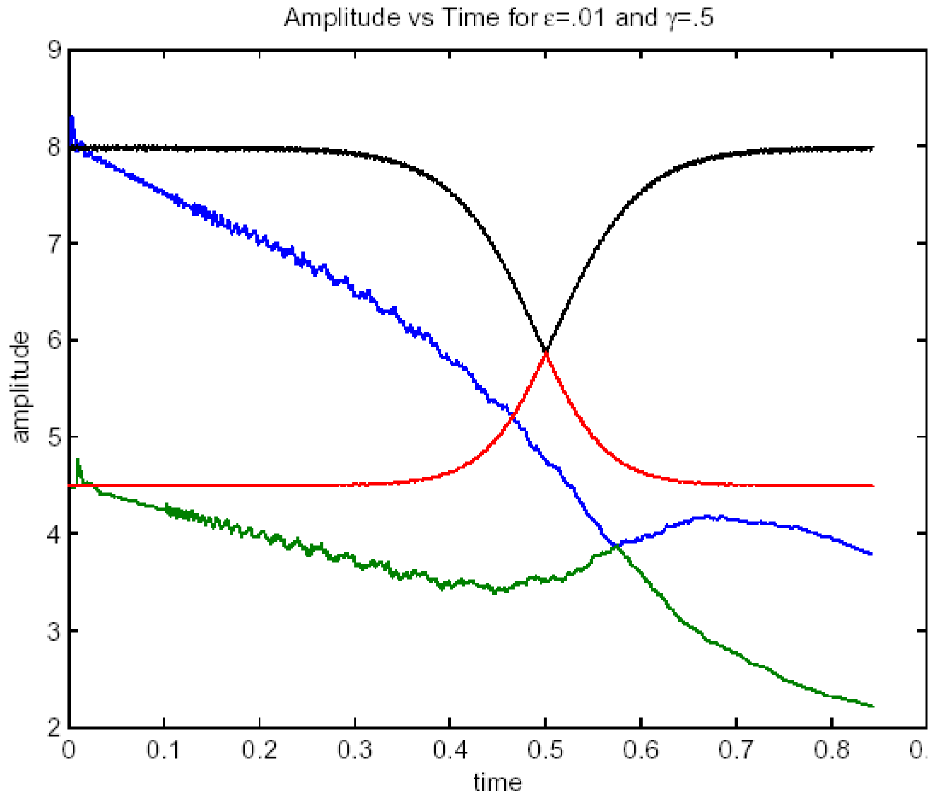

Two Soliton solution for ε=.5 and γ=.01
amplitude
10
5
0
-5
time
1
0.8
0.6
0.4
0.2
0
x
-10
0
10
20

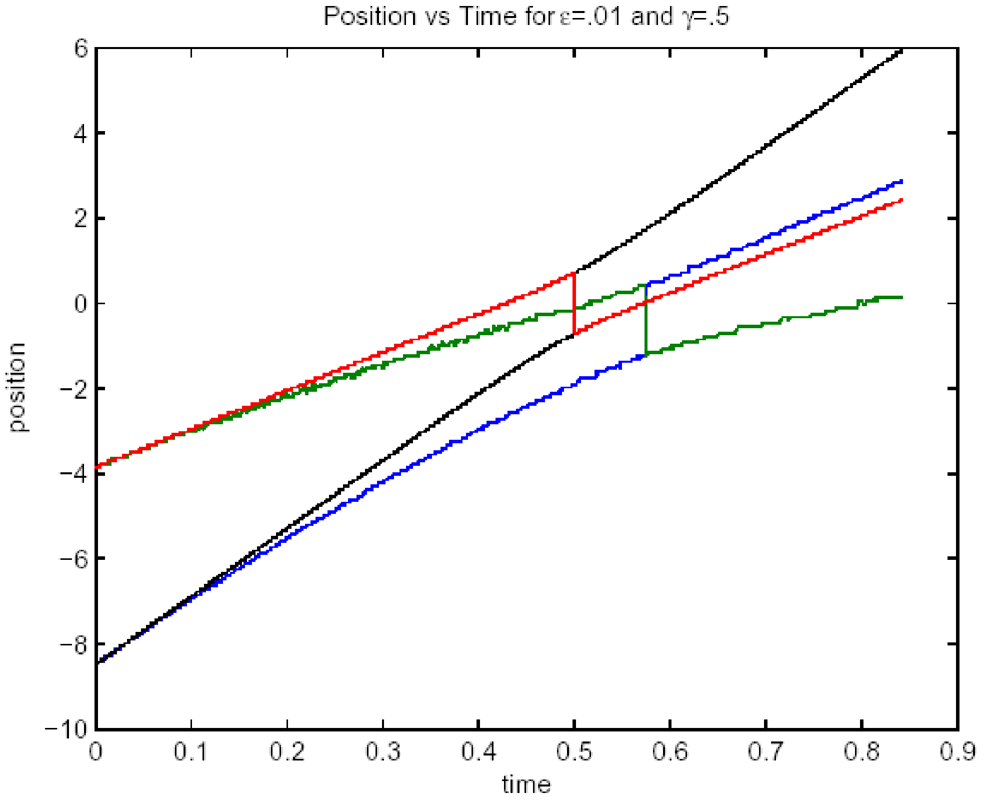
Amplitude and Position of Damped Solitons Under Noise





Amplitude and Position of Damped Solitons Under Noise





Conclusions

1. Exact Solution of Stochastic KdV
2. Damped Stochastic KdV
3. Asymptotic Results - Theory
4. Numerical Solution of Damped Stochastic KdV
5. Asymptotic Results - Numerical

References

[1] R. L. Herman, “Solitary Waves,” *American Scientist*, vol. 80, July-August 1992, 350-361.

[2] R. L. Herman and C. J. Knickerbocker, “Numerically Induced Phase Shift in the KdV Soliton,” *Journal of Computational Physics*, vol. 104, no. 1, January 1993, 50-55.

[3] Desmond J. Higham, “An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations,” *SIAM Review*, vol. 43, no. 3, 2001, 525-546.

[4] A. C. Vliegenthart, “On Finite-Difference Methods for the Korteweg-de Vries Equation,” *Journal of Engineering Mathematics*, vol. 5, no. 2, April 1971, 137-155.

[5] Miki Wadati, “Stochastic Korteweg-de Vries Equation,” *Journal of the Physical Society of Japan*, vol. 52, no. 8, August 1983, 2642-2648.

[6] Miki Wadati and Yasuhiro Akutsu, “Stochastic Korteweg-de Vries Equation with and without Damping,” *Journal of the Physical Society of Japan*, vol. 53, no. 10, October 1984, 3342-3350.