“Can One Hear the Shape of a Drum?”

Keynote for 2017 Student Award Ceremony

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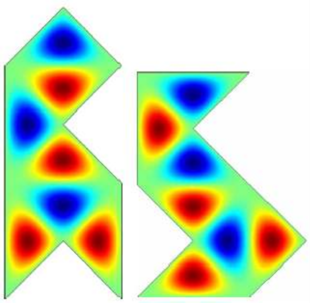

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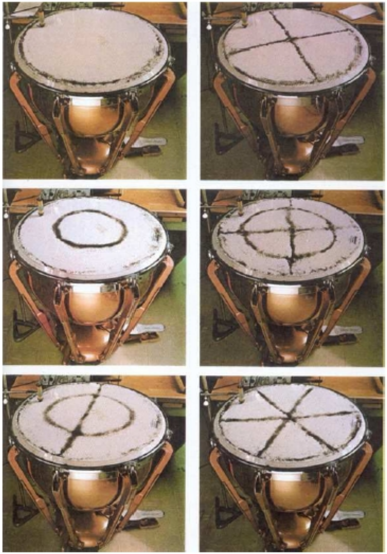
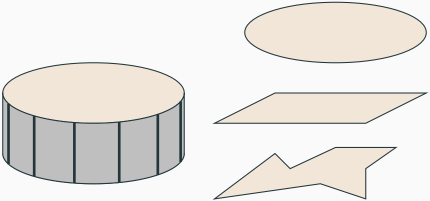


Chladni patterns on a kettledrum
from Risset, *Les instruments de l'orchestre*

Introduction

“Can One Hear the Shape of a Drum?”

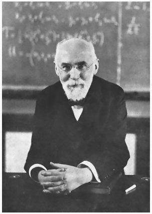
- Kac, Mark (1966). Amer. Math. Monthly. 73, Part II: 1–23.
- Title due to Lipman Bers: “If you had perfect pitch, could you hear the shape of a drum?”
- Can the frequencies (**eigenvalues**) of a resonator (**drum**) determine its shape (**geometry**)?
- Entails features of applied mathematics.
- Historical connections - from radiation theory.



Radiation Theory

- Hendrik Lorentz's (1910) 5 lectures on old/new physics problems
- 4th - Electromagnetic Radiation Theory
- Compared to organ pipe
- The number of overtones in frequency range is independent of shape, proportional to volume
- David Hilbert's prediction
- Hermann Weyl - < 2 yrs

$$N(\lambda) = \sum_{\lambda_n < \lambda} \sim \frac{|\Omega|}{2\pi} \lambda$$



What Do We Hear? Frequency, $f = \omega/2\pi$,

- Seek Harmonic Solutions,

$$u(\mathbf{r}, t) = U(\mathbf{r})e^{i\omega t},$$

- of a Wave Equation, $u(\mathbf{r}, t)$

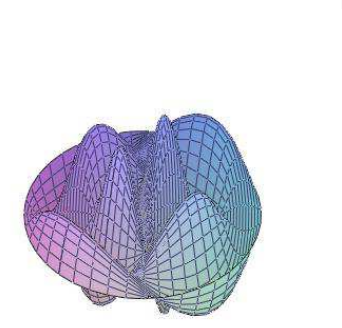
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

- Helmholtz Equation

$$\nabla^2 U = -\lambda U$$

- Eigenvalues $\sim$ frequencies

$$\lambda = \frac{\omega}{c} = k^2$$



Strings

Vibrations of a String

- Ex: Violin String.
- Harmonics, $u_n(x)$.
- Wavelength, $\lambda = \frac{2L}{n}$
- Wave Speed, $c = \sqrt{\frac{T}{\mu}}$
- Frequency, $f = n\frac{c}{2L}$
- A - $f = 440$ Hz, $L = 32$ cm.
  $c = 2Lf = 280$ m/s.
- Nodes, $u_n(x) = 0$

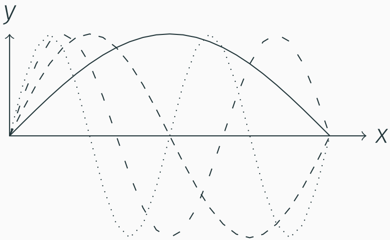


Figure 1: Plot of the eigenfunctions $u_n(x) = \sin\frac{n\pi x}{L}$ for $n = 1, 2, 3, 4$.

Solution of 1D Wave Equation

The one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \leq x \leq L, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 < x < L.$$

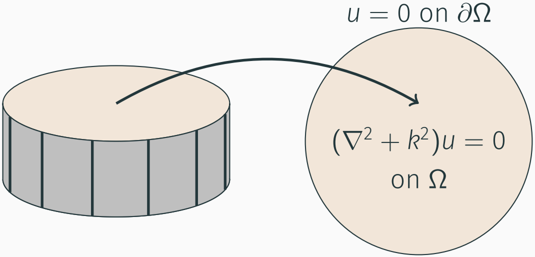
$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right] \sin \frac{n\pi x}{L}, \tag{2}$$

where $\omega_n = \frac{n\pi c}{L}$

Membranes

General 2D Membranes

- Membrane Problems.
  - Rectangular
  - Circular
  - Elliptical
  - Irregular
- Solve Helmholtz Equations
  - Normal Modes and Frequencies of Oscillation
  - Eigenvalues of Laplace Operator, $\nabla^2 u = -\lambda u$.



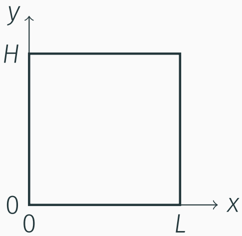
$u = 0$ on $\partial\Omega$

$(\nabla^2 + k^2)u = 0$ on Ω

Vibrations of a Rectangular Membrane

- Harmonics
- Frequencies

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$



Boundary-value problem

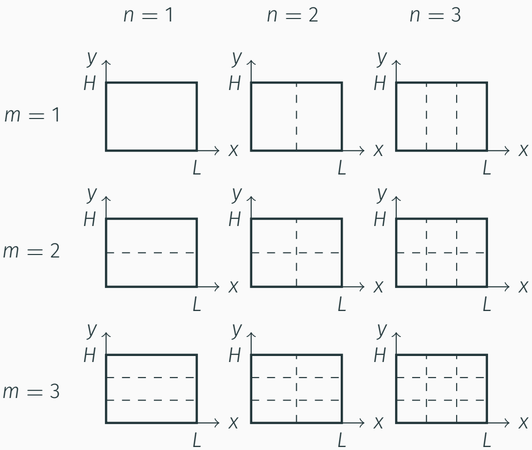
$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{3}$$

$$u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad t > 0, \quad 0 < y < H,$$
$$u(x, 0, t) = 0, \quad u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L,$$

$$u(x, y, t) = \sum_{n,m}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

Nodes of a Rectangular Membrane

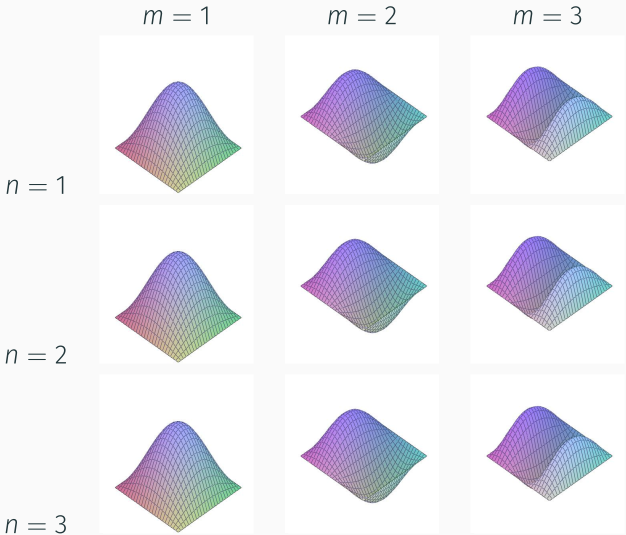
$$u_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad f = \frac{c}{2L}\sqrt{n^2+\alpha^2 m^2}, \quad \alpha = \frac{L}{H}.$$



$\alpha = 1$	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

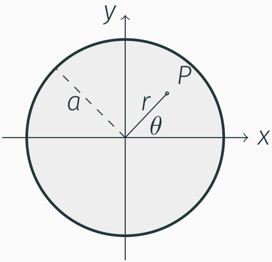
$\alpha = 2$	1	2	3
1	2.236	4.123	6.083
2	2.828	4.472	6.325
3	3.606	5.000	6.708

Vibrations of a Rectangular Membrane



Vibrations of a Circular Membrane

- Circular Symmetry.
- Harmonics
- Frequencies



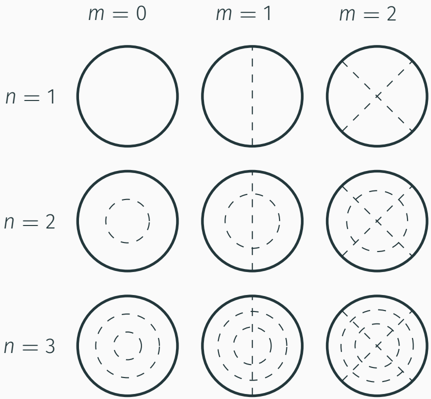
$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{j_{mn}}{a} r\right). \tag{4}$$

$$J_m(j_{mn}) = 0 \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots .$$

Nodes of a Circular Membrane

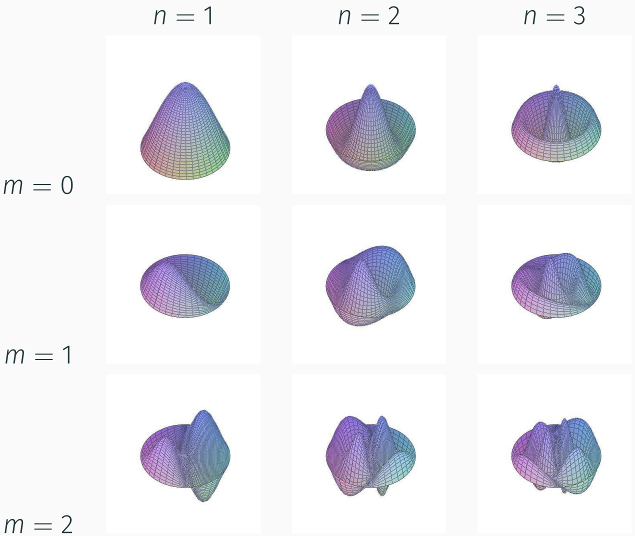
$$u_{mn}(r,\theta) = J_m\left(\frac{j_{mn}}{a}r\right)\cos m\theta, \quad f_{mn} = \frac{j_{mn}c}{2\pi a}.$$



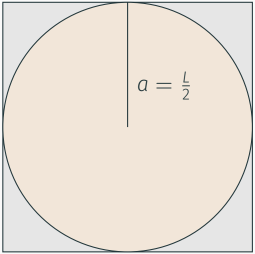
j_{mn}	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	10.173	11.62

f_{mn}	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Vibrations of a Circular Membrane



Rectangular and Circular Membrane Frequencies



Rectangular

	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

Circular $a = \frac{L}{2}$

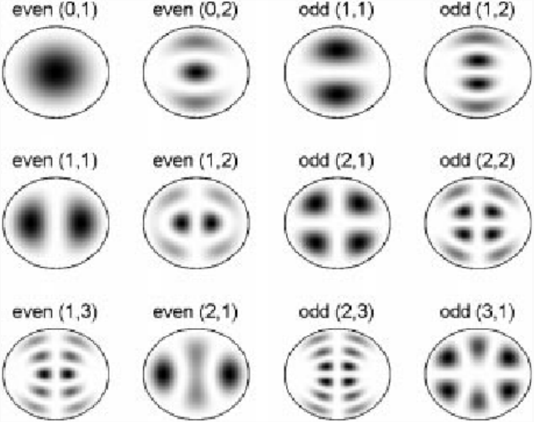
	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Circular $\pi a^2 = L^2$

	0	1	2
1	1.357	2.162	2.898
2	3.114	3.958	4.749
3	4.882	5.740	6.556

Vibrations of an Elliptical Membrane

$$\left[\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+(kh)^2(\cosh^2\xi-\cos^2\eta)\right]u(\xi,\eta)=0.$$



Vibrations of a Balloon

The wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} Lu, \quad \text{where} \quad LY_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$, The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} \cos \omega_\ell t + B_{\ell m} \sin \omega_\ell t\right] Y_{\ell m}(\theta, \phi),$$

where $\omega_\ell = \sqrt{\ell(\ell+1)} \frac{c}{R}$.

Modes for a Vibrating Spherical Membrane:

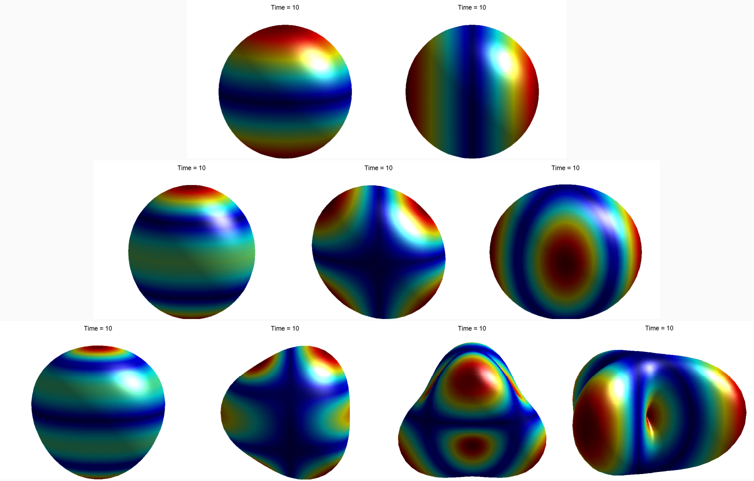
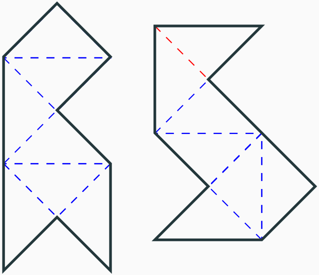


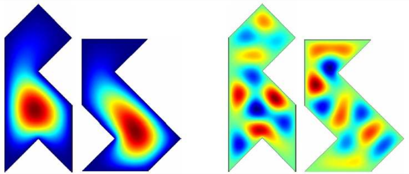
Figure 2: `http://people.uncw.edu/hermanr/pde1/sphmem/`
Row 1: $(1,0), (1,1)$; Row 2: $(2,0), (2,1), (2,2)$;
Row 3 $(3,0), (3,1), (3,2), (3,3)$.

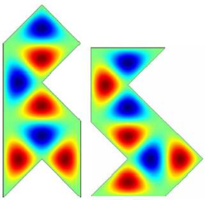
Answer

Vibrations of a Irregular Membranes

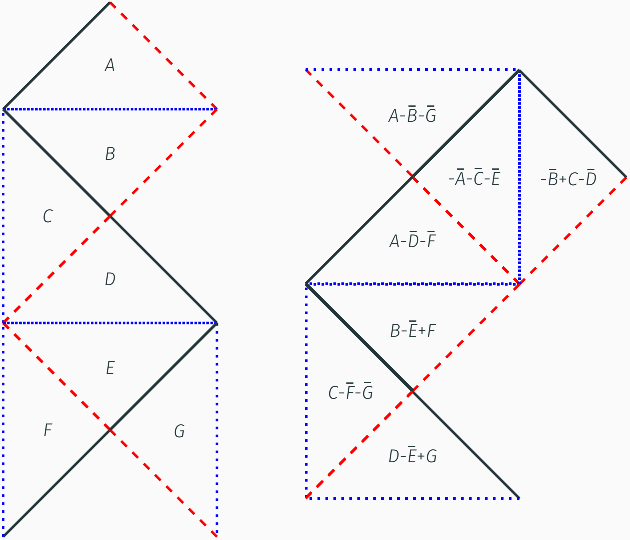


- Gordon, C., Webb, D., and Wolpert, S.(1992) - *You Cannot Hear the Shape of a Drum*
- Shapes on right have same set of frequencies - **isospectral drums.**



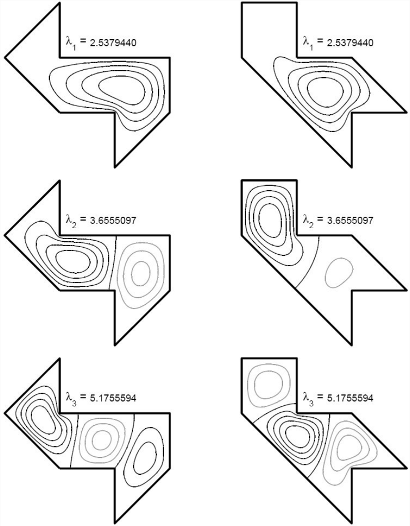


Isospectral Drums



Spectra of Isospectral Drums

$\lambda = 2.5379440, 3.6555097, 5.1755594.$



Other Isospectral Drums

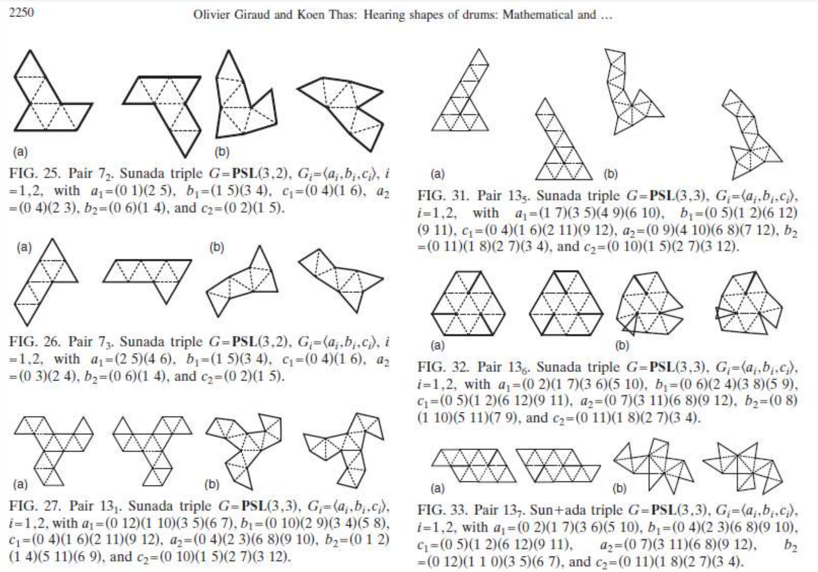


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FIG. 25. Pair 7_2. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 1)(2\ 5)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 4)(2\ 3)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

FIG. 26. Pair 7_3. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(2\ 5)(4\ 6)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 3)(2\ 4)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

FIG. 27. Pair 13_1. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 12)(1\ 10)(3\ 5)(6\ 7)$, $b_1=(0\ 10)(2\ 9)(3\ 4)(5\ 8)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $b_2=(0\ 1\ 2)(1\ 4)(5\ 11)(6\ 9)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

FIG. 31. Pair 13_5. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(1\ 7)(3\ 5)(4\ 9)(6\ 10)$, $b_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 9)(4\ 10)(6\ 8)(7\ 12)$, $b_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

FIG. 32. Pair 13_6. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 6)(2\ 4)(3\ 8)(5\ 9)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 8)(1\ 10)(5\ 11)(7\ 9)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

FIG. 33. Pair 13_7. Sun+ada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 12)(1\ 1\ 0)(3\ 5)(6\ 7)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

Summary

Summary

- Can one hear the shape of a drum? - No!
- Membranes - Rectangular, circular, elliptical, irregular
- Never look at MATLAB logo the same way again - Why?

References I

- S. J. Chapman, Drums that sound the same, *Amer. Math. Monthly* 102 (1995), 124-138.
- Tobin Driscoll, Eigenmodes of isospectral drums, SIAM Review 39 (1997), 1-17.
- Carolyn Gordon, David Webb, Scott Wolpert, One cannot hear the shape of a drum, *Bull. Amer. Math. Soc.* 27 (1992), 134-138.
- Marc Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* 73 (1966), 1-23.
- Cleve Moler, The MathWorks logo is an eigenfunction of the wave equation (2003).
- Lloyd N. Trefethen and Timo Betcke, Computed eigenmodes of planar regions (2005).

Thank You!

Thank you for your time … and now on with the awards!