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Revisiting Quasistationary Perturbation Theory for Equations in 1+1 Dimensions

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Abstract

We revisit quasistationary perturbation theory for integrable systems and compare the solutions to those obtained through eigenfunction expansion methods. We will focus our comparisons to the perturbed nonlinear equations such as the Korteweg de Vries, the nonlinear Schrodinger and the sine-Gordon equations.

1. History
2. Soliton Perturbation Theory
3. Quasistationary Perturbations
4. Direct Integration - KdV
5. Eigenfunction Expansions - KdV
6. Comparison of Results
7. Perturbed NLS

1 History

1. Ott and Sudan 1969
2. Karpman and Maslov 1977 IST
3. Kaup and Newell 1977, Kaup 1976 IST
4. Keener and McLaughlin 1977 Green's Function Method
5. Kodama and Ablowitz 1981 Quasistationary
6. Sachs 1984 IST
7. Menyuk 1986 Direct
8. Kivshar and Malomed 1989 Review
9. Herman 1990 Direct + IST
10. Yan and Tang 1996 Direct + IST
11. Mann 1997 Green's Function Method
12. and many others in soliton perturbation theory

2 Soliton Perturbation Theory

$$u_T + \mathcal{N}[u] = \epsilon R[u], \qquad 0 < \epsilon \ll 1. \tag{1}$$

Multiple Scales

$$\partial_T = \partial_t + \epsilon \partial_\tau$$

Expansion about soliton solution $u_0 = u_0(z, \tau)$, $z = x - vt$

$$u(x, T) = u_0(z, \tau) + \epsilon u_1(z, t, \tau) + \dots. \tag{2}$$

Linearized Equation

$$u_{1t} + \hat{L}[u_1] = R[u_0] - u_{0\tau} = F(z). \tag{3}$$

Eigenfunction Expansion Method

Eigenfunctions and Adjoint functions

$$\hat{L}\phi = \lambda\phi, \qquad \hat{L}^\dagger\psi = \lambda'\psi. \tag{4}$$

Expansion in eigenfunctions

$$u_1(z, t) = \int U(t, \lambda)\phi(z, \lambda)\, d\lambda + \sum_j U_j(t)\phi_j(z). \tag{5}$$

$$U_t + \lambda U, = \int_{-\infty}^{\infty} F(z)\psi(z, \lambda)\, dz., \quad U(0, \lambda) = 0, \tag{6}$$

3 Example: KdV Equation

The perturbed KdV equation

$$u_T + 6uu_x + u_{xxx} = \epsilon R[u], \tag{7}$$

Leading Order:

$$u_0(z) = 2\eta^2 \operatorname{sech}^2 z, \qquad z = \eta(x - \xi), \text{ and } \xi_t = 4\eta^2$$

.

First Order:

$$u_{1t} + \eta^3 \hat{L} u_1 = R[u_0] - 4\eta\eta_\tau \phi_1(z) - 4\eta^3 \xi_\tau \phi_2(z) \equiv F(z), \tag{8}$$

where

$$\begin{aligned} \phi_1(z) &= (1 - z \tanh z) \operatorname{sech}^2 z, \\ \phi_2(z) &= \operatorname{sech}^2 z \tanh z. \end{aligned} \tag{9}$$

and

$$\hat{L} = \frac{d^3}{dz^3} + (12 \operatorname{sech}^2 z - 4) \frac{d}{dz} - 24 \operatorname{sech}^2 z \tanh z. \tag{10}$$

Quasistationarity assumption: $u_1(z,t) = u_1(z)$

$$\eta^3 \hat{L} u_1 = F(z). \tag{11}$$

3.1 Direct Integration: Kodama and Ablowitz [5]

$$\eta^3 \hat{L} u_1 = F(z). \tag{12}$$

Set $y = \tanh(z)$, $g(y) = u_1(z)$ and $\tilde{F}(y) = F(z)$.

Rearranging and integrating,

$$\frac{d}{dy}\left((1-y^2)\frac{dg}{dy}\right) + (12 - \frac{4}{1-y^2})g = \frac{1}{1-y^2}\int \frac{\tilde{F}(y)}{\eta^3(1-y^2)}\,dy. \tag{13}$$

The Method of Variation of Parameters: $g(y) = A(y)P_3^2(y) = A(y)15y(1-y^2)$

$$15y(1-y^2)^2\left[\frac{d^2A}{dy^2} + \frac{2(1-4y^2)}{y(1-y^2)}\frac{dA}{dy}\right] = \mathcal{F}(y), \tag{14}$$

where

$$\mathcal{F}(y) \equiv \frac{1}{1-y^2}\int \frac{\tilde{F}(y)}{\eta^3(1-y^2)}\,dy. \tag{15}$$

Solve for $A(y)$:

$$A(y) = \int^y dx \frac{1}{x^2(1-x^2)^3}\int^x dw \frac{1}{15} w(1-w^2)\mathcal{F}(w), \tag{16}$$

Leads to the general solution

$$u_1(z) = \left[\frac{y(1-y^2)}{\eta^3}\int^y dx\,\frac{1}{x^2(1-x^2)^3}\int^x dw\,w\int^w ds\,\frac{\tilde{F}(s)}{1-s^2}\right]_{y=\tanh z}. \tag{17}$$

3.2 The Damped KdV Equation

$$u_t + 6uu_x + u_{xxx} = -\epsilon\gamma u. \tag{18}$$

$$F(z) = -2\eta\left(\eta\gamma + 2\eta_\tau - 2\eta_\tau z\tanh z + 2\eta^2\xi_\tau\tanh z\right)\operatorname{sech}^2 z. \tag{19}$$

Compatibility condition:

$$\int_{-\infty}^{\infty} F(z)\operatorname{sech}^2 z\, dz = 0 \Rightarrow \eta_\tau = -\frac{2}{3}\eta\gamma. \tag{20}$$

The forced, linearized problem

$$\eta^3\hat{L}u_1 = \frac{2}{3}\gamma\eta^2\operatorname{sech}^2 z(1 - 4z\tanh z) - 4\eta^3\xi_\tau\operatorname{sech}^2 z\tanh z. \tag{21}$$

Apply the general solution to obtain

$$\begin{aligned} u_1(z) &= \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z \\ &\quad + C_1\operatorname{sech}^2 z\tanh z + C_2(-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z), \end{aligned} \tag{22}$$

The general solution to $\hat{L}v = 0$:

$$v(z) = C_1\operatorname{sech}^2 z\tanh z + C_2[-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z] + C_3\cosh^2 z.$$

3.3 Eigenfunction Expansion: Yan and Tang, Phys Rev E 54, 1996

First Order

$$u_{1t} + \eta^3 \hat{L} u_1 = R_1 - 4\eta\eta_\tau \phi_1(z) - 4\eta^3 \xi_\tau \phi_2(z)$$
$$u_0(z) = 2\eta^2 \operatorname{sech}^2 z, \qquad z = \eta(x - \xi) \tag{23}$$

Linear operator and adjoint operator:

$$\hat{L} = \frac{d^3}{dz^3} + (12 \operatorname{sech}^2 z - 4)\frac{d}{dz} - 24 \tanh z \operatorname{sech}^2 z \tag{24}$$

$$\hat{L}^\dagger = \frac{d^3}{dz^3} + (12 \operatorname{sech}^2 z - 4)\frac{d}{dz} \tag{25}$$

Eigenfunctions:

$$\hat{L}\phi = \lambda\phi, \qquad \lambda = -ik(k^2 + 4)$$
$$\hat{L}^\dagger \psi = \lambda' \psi, \qquad \lambda' = ik(k^2 + 4) \tag{26}$$

Continuous States

$$\phi(z,k) = \frac{1}{\sqrt{2\pi} k(k^2+4)} \left[k(k^2+4) + 4i(k^2+2)\tanh z - 8k \tanh^2 z - 8i \tanh^3 z\right] e^{ikz},$$
$$\psi(z,k) = \frac{1}{\sqrt{2\pi}(k^2+4)} \left[k^2 - 4ik \tanh z - 4\tanh^2 z\right] e^{-ikz}. \tag{27}$$

Bound (discrete) states

$$\phi_1(z) = (1 - z\tanh z)\operatorname{sech}^2 z, \qquad \phi_2(z) = \tanh z \operatorname{sech}^2 z,$$
$$\psi_1(z) = \operatorname{sech}^2 z, \qquad \psi_2(z) = \tanh z + z \operatorname{sech}^2 z. \tag{28}$$

Perturbation Expansion

Completeness Relations and Orthogonality

$$P\int_{-\infty}^{\infty}\phi(z,k)\psi(z',k)\,dk+\sum_{j=1}^{2}\phi_j(z)\psi_j(z')=\delta(z-z'), \tag{29}$$

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(z,k)\psi(z,k')\,dz &= \delta(k-k'),\\ \int_{-\infty}^{\infty}\phi_j(z)\psi_\ell(z)\,dz &= \delta_{j,\ell}, \qquad j,k=1,2\end{aligned} \tag{30}$$

Thus, expansions are of the form

$$\begin{gathered}F(z)=P\int_{-\infty}^{\infty}f(k)\phi(z,k)\,dk+\sum_{j=1}^{2}f_j\phi_j(z),\\ f(k)=\int_{-\infty}^{\infty}F(z)\psi(z,k)\,dz, \qquad f_j=\int_{-\infty}^{\infty}F(z)\psi_j(z)\,dz, \qquad j=1,2\end{gathered} \tag{31}$$

$$u_1(z,t)=P\int_{-\infty}^{\infty}U(t,k)\phi(z,k)\,dk+\sum_{j=1}^{2}U_j(t)\phi_j(z) \tag{32}$$

Then, for $u_{1t}+\eta^3\hat{L}u_1=F$

$$\begin{aligned}U_t+\eta^3\lambda(k)U, &= f(k), \quad U(0,k)=0,\\ U_{1t} &= f_1, \quad U_1(0)=0,\\ U_{2t}-8\eta^3U_1 &= f_2, \quad U_2(0)=0.\end{aligned} \tag{33}$$

Quasistationary Damped KdV - Eigenfunction Expansion

Problem:

$$\eta^3 \hat{L} u_1 = -2\eta^2 \gamma \operatorname{sech}^2 z - 4\eta\eta_\tau \phi_1(z) - 4\eta^3 \xi_\tau \phi_2(z) \equiv F(z) \tag{34}$$

$$\hat{L}\phi = \lambda\phi, \qquad \lambda = -ik(k^2+4) \tag{35}$$

Let

$$u_1(z) = P\int_{-\infty}^{\infty} U(k)\phi(z,k)\,dk + U_1\phi_1(z) + U_2\phi_2(z) \tag{36}$$

Then

$$\hat{L}u_1 = P\int_{-\infty}^{\infty} U(k)[-ik(k^2+4)]\phi(z,k)\,dk - 8U_1\phi_2(z) \tag{37}$$

Expand $F(z)$.

$$F(z) = P\int_{-\infty}^{\infty} f(k)\phi(z,k)\,dk + f_1\phi_1(z) + f_2\phi_2(z) \tag{38}$$

Equating coefficients

$$U(k)\eta^3[-ik(k^2+4)] = f(k) = \int_{-\infty}^{\infty} F(z)\psi(z,k)\,dx \tag{39}$$

$$0 = f_1 = \int_{-\infty}^{\infty} F(z)\operatorname{sech}^2 z\,dz \tag{40}$$

$$-8\eta^3 U_1 = f_2 = \int_{-\infty}^{\infty} F(z)[\tanh z + z\operatorname{sech}^2 z]\,dz \tag{41}$$

where

$$\psi(z,k) = \frac{1}{\sqrt{2\pi}k(k^2+4)}\left[k^2 - 4ik\tanh z - 4\tanh^2 z\right]e^{-ikz} \tag{42}$$

Damped KdV Results

Expansion coefficients:

$$f(k) = \frac{\sqrt{2\pi}}{3}\frac{\gamma\eta^2 k}{\sinh(\frac{\pi k}{2})}, \tag{43}$$

$$f_1 = -\frac{8}{3}\gamma\eta^2 - 4\eta\eta_\tau, \qquad f_1 = 0 \Rightarrow \eta_\tau = -\frac{2}{3}\eta\gamma \tag{44}$$

$$f_2 = -4\eta^3\xi_\tau. \tag{45}$$

Sum over the continuum states:

$$I = P\int_{-\infty}^{\infty} U(k)\phi(z,k)\,dk, \tag{46}$$

$$U(k) = -\frac{\sqrt{2\pi}}{3i\eta\gamma}\frac{1}{(k^2+4)\sinh(\frac{\pi k}{2})}. \tag{47}$$

Inserting $\phi(z,k)$ yields

$$I = \frac{i\gamma}{3\eta}P\int_{-\infty}^{\infty}\frac{k(k^2+4)+4i(k^2+2)\tanh z - 8k\tanh^2 z - 8i\tanh^3 z}{k(k^2+4)^2\sinh(\frac{\pi k}{2})}\;e^{ikz}\,dk. \tag{48}$$

The result:

$$I = \frac{\gamma}{6\eta}\left[-\left(\frac{\pi^2}{12}+z^2+\frac{3}{2}\right)\operatorname{sech}^2 z\tanh z + \tanh z + 2z\operatorname{sech}^2 z\right]. \tag{49}$$

The full solution of the quasistationary problem:

$$\begin{aligned} u_1 \quad &= \quad \frac{\gamma}{6\eta}\left[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z\right] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z \\ &\quad +\tilde{C}\operatorname{sech}^2 z\tanh z. \end{aligned} \tag{50}$$

4 Evaluation of Integral (48)

$$I = \frac{i\gamma}{3\eta} P \int_{-\infty}^{\infty} \frac{k(k^2+4) + 4i(k^2+2)\tanh z - 8k\tanh^2 z - 8i\tanh^3 z}{k(k^2+4)^2 \sinh(\frac{\pi k}{2})} \, e^{ikz} \, dk. \tag{51}$$

Consider the evaluation of the integrals:

$$I_1 = P \int_{-\infty}^{\infty} \frac{e^{ikz}}{(k^2+4)\sinh(\frac{\pi k}{2})} \, dk. \tag{52}$$

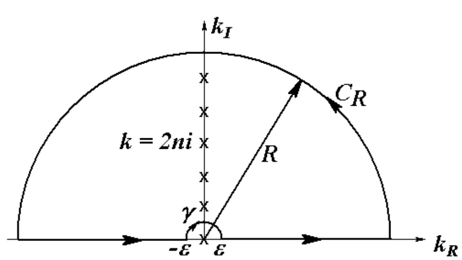
$$I_2 = P \int_{-\infty}^{\infty} \frac{e^{ikz}}{k(k^2+4)\sinh(\frac{\pi k}{2})} \, dk. \tag{53}$$

$$I_3 = P \int_{-\infty}^{\infty} \frac{e^{ikz}}{(k^2+4)^2\sinh(\frac{\pi k}{2})} \, dk. \tag{54}$$

$$I_4 = P \int_{-\infty}^{\infty} \frac{e^{ikz}}{k(k^2+4)^2\sinh(\frac{\pi k}{2})} \, dk. \tag{55}$$

Then the full solution would be given by

$$I = \frac{i\gamma}{3\eta} \left[I_1 + 4i(I_2 - 2I_4)\tanh z - 8I_3 \tanh^2 z - 8iI_4 \tanh^3 z \right]. \tag{56}$$



Computation of I_m's

For $z > 0$,

$$I_m = 2\pi i \sum_{n=1}^{\infty} \mathrm{Res}[f_m(k_n); k_n = 2in] + \pi i \mathrm{Res}[f_m(k); k = 0]. \tag{57}$$

We find that

$$I_1 = \frac{i}{2} - \frac{i}{4}(4z+1)e^{-2z} - i \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} e^{-2nz}, \tag{58}$$

$$I_2 = -\frac{z}{2} - \frac{1}{8}(4z+3)e^{-2z} - \frac{1}{4} \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n^2-1)} e^{-2nz}, \tag{59}$$

$$I_3 = \frac{i}{8} - \frac{i}{192}(24z^2 + 24z + 2\pi^2 + 9)e^{-2z} + \frac{i}{4} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n^2-1)^2} e^{-2nz}, \tag{60}$$

$$I_4 = -\frac{z}{8} - \frac{1}{384}(24z^2 + 48z + 2\pi^2 + 33)e^{-2z} + \frac{1}{8} \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n^2-1)^2} e^{-2nz}. \tag{61}$$

Sum infinite series using partial fraction decomposition and the summations

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{-2nz}}{n} = \ln(1 + e^{-2z}), \tag{62}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{-2nz}}{n^2} = \mathrm{Li}_2(e^{-2z}) \equiv \int_0^{e^{-2z}} \frac{\ln(1+x)}{x}\, dx. \tag{63}$$

For $z < 0$ one obtains similar results

$$I_1 = \frac{i}{2} + \frac{i}{4}(4z-1)e^{2z} - i \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} e^{2nz}, \tag{64}$$

4.1 Comparison of Quasistationary Results

For the damped KdV equation,

$$\eta^3 \hat{L} u_1 = \frac{2}{3}\gamma\eta^2 \operatorname{sech}^2 z(1 - 4z\tanh z) - 4\eta^3 \xi_\tau \operatorname{sech}^2 z \tanh z, \tag{65}$$

Direct integration method:

$$\begin{aligned} u_1(z) &= \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z \\ &\quad + C_1 \operatorname{sech}^2 z \tanh z + C_2(-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z). \end{aligned} \tag{66}$$

Eigenfunction expansion method:

$$\begin{aligned} u_1 &= \frac{\gamma}{6\eta}\left[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z\right] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z \\ &\quad + \tilde{C} \operatorname{sech}^2 z \tanh z. \end{aligned} \tag{67}$$

The same compatibility condition: $\eta_\tau = -\frac{2}{3}\eta\gamma$.

However, these solutions differ by a solution of the homogeneous equation, $v(z) \equiv -1 + 3(1 - z\tanh z)\operatorname{sech}^2 z$.

We can write $v(z) = -1 + 3(1 - z\tanh z)\operatorname{sech}^2 z = -1 + 3\phi_1(z)$. Noting

$$1 = P\int_{-\infty}^{\infty} h(k)\phi(z,k)\,dk + 2\phi_1(z), \tag{68}$$

$$v(z) = -P\int_{-\infty}^{\infty} h(k)\phi(z,k)\,dk + \phi_1(z). \tag{69}$$

where $h(k) = \frac{k^2-4}{k^2+4}\frac{\delta(k)}{\sqrt{\pi}}$. Therefore, the $k = 0$ pole contributes to this particular solution of the homogeneous problem.

Modification of the Eigenfunction Expansion Method

We first note that

$$\hat{L}[1] = -24\phi_2$$

and recall that

$$\hat{L}\phi_1 = -8\phi_2.$$

We seek a solution of $\hat{L}v = 0$ by considering $v = 1 + \beta\phi_1$.

Then

$$0 = \hat{L}[1 + \beta\phi_1] = (-24 - 8\beta)\phi_2.$$

Therefore, we recover the missing solution $v(z) = 1 - 3\phi_1(z)$.

Comparison with Kodama and Ablowitz [5]

They gave the solution for $u_1(z)$ as

$$u_1(z) = \frac{\gamma}{6\eta}[-1 + \tanh z + 3\left(1 + \frac{\eta}{\gamma}\xi_\tau\right)(1 - z\tanh z)\operatorname{sech}^2 z + z(2 - z\tanh z)\operatorname{sech}^2 z]. \tag{70}$$

Agrees with (22) up to terms proportional to $\operatorname{sech}^2 z\tanh z$. In fact, this $u_1(z)$ can be rewritten as

$$\begin{aligned} u_1(z) &= \frac{\gamma}{6\eta}[\tanh z + z(2 - z\tanh z)\operatorname{sech}^2 z] + \frac{1}{2}\xi_\tau(1 - z\tanh z)\operatorname{sech}^2 z \\ &\quad + \frac{\gamma}{6\eta}(-1 + 3(1 - z\tanh z)\operatorname{sech}^2 z), \end{aligned} \tag{71}$$

showing that the extra term is proportional to $v(z)$ above. The constant $C_2 = \frac{\gamma}{6\eta}$ from the general solution (66) can be obtained by imposing the boundary condition $u_1(z) \to 0$ as $z \to \infty$.

5 Loose Ends

Secularity Conditions?

In both problems we arrived at the condition

$$\int_{-\infty}^{\infty} F(z)\,\mathrm{sech}^2 z\,dz = 0. \tag{72}$$

In the non-quasistationary perturbation theory, there is a second condition, which gives the correction to the soliton velocity

$$\int_{-\infty}^{\infty} F(z)[\tanh z + z\,\mathrm{sech}^2 z + \tanh^2 z]\,dz = 0. \tag{73}$$

Use conservation of energy. Namely,

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx = -2\epsilon\gamma\int_{-\infty}^{\infty} u^2\,dx. \tag{74}$$

Then

$$\xi_\tau = -\frac{\gamma}{3\eta}.$$

Arbitrariness in the solution in the form of $\mathrm{sech}^2 z\tanh z$ terms.

Note that to order ϵ we have

$$\begin{aligned} u_0(z) &= 2\eta^2\,\mathrm{sech}^2(\eta(x-\xi+\epsilon x_0)) \\ &\approx 2\eta^2\,\mathrm{sech}^2 z - 4\epsilon x_0\eta^3\,\mathrm{sech}^2 z\tanh z. \end{aligned} \tag{75}$$

Integrating $\xi_\tau = -\frac{\gamma}{3\eta}$ while using $\eta_\tau = -\frac{2}{3}\eta\gamma$,

$$\xi = \xi_0 - \frac{1}{2}e^{2\gamma\tau/3}. \tag{76}$$

Thus, we can pick $C_1 = -4x_0\eta^3$ in order to adjust the phase of the perturbed solution at $\tau = 0$.

5.1 Solution Plots

The solution of the forced, linearized KdV equation under the quasistationary assumption:

$$u_1(z) = \frac{\gamma}{6\eta}[-1 + \tanh z + 2(1 - z\tanh z)\,\mathrm{sech}^2 z + z(2 - z\tanh z)\,\mathrm{sech}^2 z]. \tag{77}$$

The range of validity:$|z| \ll \epsilon^{-1/2}$. For $\epsilon = 0.01$ this would give $|z| \ll 10$. Kodama and Ablowitz [5]

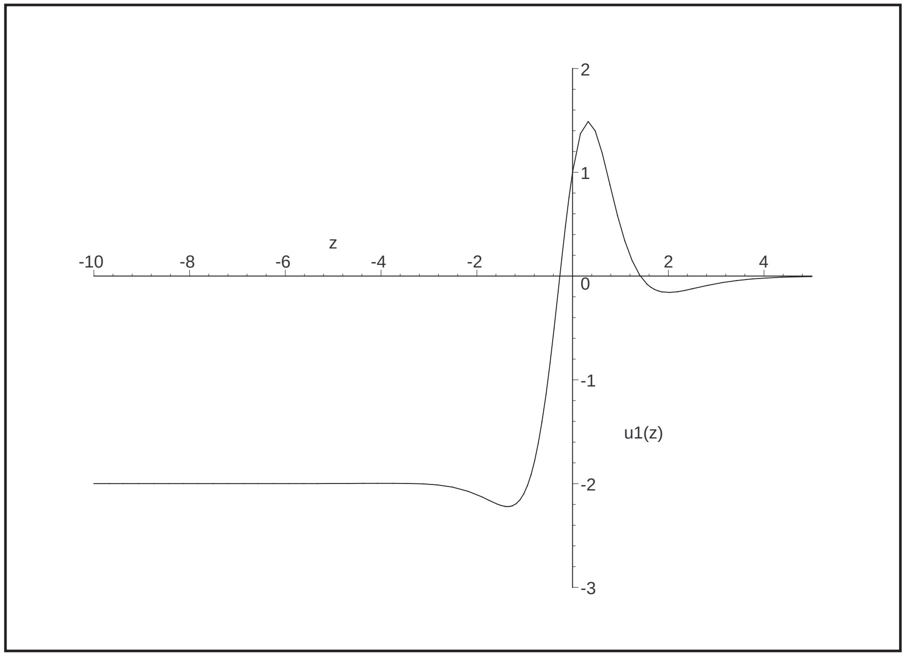


Figure 1: This is a scaled plot of the first order solution $u_1(z)$ in equation (77).

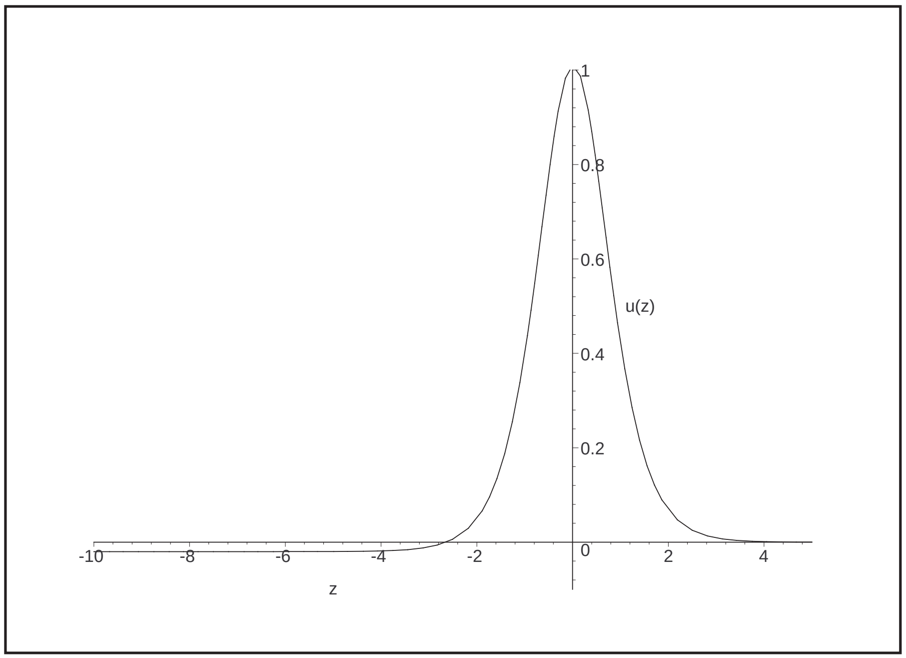


Figure 2: In this figure we plot $u(z) = u_0(z) + \epsilon u_1(z)$ for $\epsilon = 0.01$ and $\gamma = 6\eta$.

6 NLS Perturbation Theory ?????

Perturbed NLS: $iu_T + u_{xx} + 2|u|^2 u = i\epsilon R[u]$ Leading Order Solution

$$u_0(z) = 2\beta e^{-i\theta} \operatorname{sech} z \tag{78}$$

$$z = 2\beta(x - \xi), \qquad \xi_t = -4\alpha,$$

$$\theta = 2\alpha(x - \xi) + \delta, \qquad \delta_t = -4(\alpha^2 + \beta^2).$$

First Order Equation

$$iu_{1t} + 8i\alpha\beta u_{1z} + 4\beta^2 u_{1zz} + 4|u_0|^2 u_1 + 2|u_0|^2 \bar{u}_1 = iF_1, \tag{79}$$

where

$$iF_1 = iR_1[u_0] - ie^{-i\theta}\left[(4i\alpha\beta\xi_\tau - 2i\beta\delta_\tau)\phi_1(z) - 2i\alpha_\tau\phi_2(z) + 2\beta_\tau\psi_1(z) + 4\beta^2\xi_\tau\psi_2(z)\right] \tag{80}$$

$$\phi_1(z) = \operatorname{sech} z, \qquad \phi_2(z) = z \operatorname{sech} z \tag{81}$$

$$\psi_1(z) = (1 - z\tanh z)\operatorname{sech} z, \qquad \psi_2(z) = \tanh z \operatorname{sech} z \tag{82}$$

For $u_1(z,t) = e^{-i\theta}[A_1(z,t) + iB_1(z,t)]$

$$\begin{aligned} A_{1t} + 4\beta^2 \hat{L}_1 B_1 &= Re(R_1[u_0]) - \left[2\beta_\tau\psi_1(z) + 4\beta^2\xi_\tau\psi_2(z)\right] \\ B_{1t} - 4\beta^2 \hat{L}_2 A_1 &= Im(R_1[u_0]) - [(4\alpha\beta\xi_\tau - 2\beta\delta_\tau)\phi_1(z) - 2\alpha_\tau\phi_2(z)] \end{aligned} \tag{83}$$

7 NLS Perturbation Theory - Eigenfunction Expansion

Quasistationary Perturbation $u_1(z) = e^{-i\theta}[A_1(z) + iB_1(z)]$

$$\begin{aligned} +4\beta^2 \hat{L}_1 B_1 &= Re(R_1[u_0]) - \left[2\beta_\tau \psi_1(z) + 4\beta^2 \xi_\tau \psi_2(z)\right] \\ -4\beta^2 \hat{L}_2 A_1 &= Im(R_1[u_0]) - [(4\alpha\beta\xi_\tau - 2\beta\delta_\tau)\phi_1(z) - 2\alpha_\tau \phi_2(z)] \end{aligned} \tag{84}$$

Linear Operators $\hat{L}_n = \frac{d^2}{dz^2} + n(n+1)\,\mathrm{sech}^2 z - 1$. Eigenvalue Problem - J. Yan, et. al., Phys Rev 58 1064 (1998).

$$\begin{bmatrix} 0 & \hat{L}_1 \\ \hat{L}_2 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \phi \end{bmatrix} = \lambda \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \tag{85}$$

Expansions ($\lambda = -(k^2+1)$)

$$A_1 = \int_{-\infty}^{\infty} a_1(k)\psi(z,k)\,dk + \sum_{j=1}^{2} a_{1j}\psi_j(z), \tag{86}$$

$$B_1 = \int_{-\infty}^{\infty} b_1(k)\phi(z,k)\,dk + \sum_{j=1}^{2} b_{1j}\phi_j(z). \tag{87}$$

$$\phi(z,k) = \frac{1}{\sqrt{2\pi}(k^2+1)}(1 - k^2 - 2ik\tanh z)e^{ikz}, \tag{88}$$

$$\psi(z,k) = \frac{1}{\sqrt{2\pi}(k^2+1)}(-1 - k^2 - 2ik\tanh z + 2\tanh^2 z)e^{ikz}. \tag{89}$$

Expansion Coefficients

$$4\beta^2 \lambda b_1(k) = \int_{-\infty}^{\infty} Re(R_1[u_0]e^{i\theta})\bar{\phi}(z,k)\,dz, \tag{90}$$

$$-4\beta^2 \lambda a_1(k) = \int_{-\infty}^{\infty} Im(R_1[u_0]e^{i\theta})\bar{\psi}(z,k)\,dz, \tag{91}$$

$$2\beta_\tau = \int_{-\infty}^{\infty} Re(R_1[u_0]e^{i\theta})\bar{\phi}_1(z)\,dz, \tag{92}$$

$$2\alpha_\tau = -\int_{-\infty}^{\infty} Im(R_1[u_0]e^{i\theta})\bar{\psi}_2(z)\,dz, \tag{93}$$

$$8\beta^2 b_{12} = 4\beta^2 \xi_\tau - \int_{-\infty}^{\infty} Re(R_1[u_0]e^{i\theta})\bar{\phi}_2(z)\,dz, \tag{94}$$

$$8\beta^2 a_{11} = 4\alpha\beta\xi_\tau - 2\beta\delta_\tau - \int_{-\infty}^{\infty} Im(R_1[u_0]e^{i\theta})\bar{\psi}_1(z)\,dz, \tag{95}$$

Damped NLS Example

Damped NLS: $iu_T + u_{xx} + 2|u|^2 u = -i\epsilon u$

$$\beta_\tau = -2\beta, \qquad \alpha_\tau = 0.$$

$$a_1(k) = 0, \qquad 4\beta^2\lambda b_1(k) = \sqrt{2\pi}\beta\,\mathrm{sech}\,(\pi k/2).$$

8 NLS Perturbation Theory - Direct Integration

Recall: Quasistationary Perturbation $u_1(z) = e^{-i\theta}[A_1(z) + iB_1(z)]$

$$\begin{aligned} +4\beta^2 \hat{L}_1 B_1 &= Re(R_1[u_0]) - \left[2\beta_\tau \psi_1(z) + 4\beta^2 \xi_\tau \psi_2(z)\right] \\ -4\beta^2 \hat{L}_2 A_1 &= Im(R_1[u_0]) - [(4\alpha\beta\xi_\tau - 2\beta\delta_\tau)\phi_1(z) - 2\alpha_\tau \phi_2(z)] \end{aligned} \tag{96}$$

where $\hat{L}_n = \frac{d^2}{dz^2} + n(n+1)\,\mathrm{sech}^2 z - 1$.

$$\hat{L}_n h(z) = F(z)$$

Let $y = \tanh z$, $\qquad h(z) = g(y)$:

Then

$$(1-y^2)\mathcal{L}_n g(y) = \tilde{F}(y), \tag{97}$$

where $\mathcal{L}_n = \frac{d}{dy}(1-y^2)\frac{d}{dy} + n(n+1) - \frac{1}{1-y^2}$.

Since $\mathcal{L}_n P_n^1(y) = 0$, use Variation of Parameters: $g(y) = f(y)P_n^1(y)$.

Then one obtains

$$\frac{d}{dy}\left[\left(P_n^1(y)\right)^2 (1-y^2)\frac{df}{dy}\right] = \frac{\tilde{F}(y)}{1-y^2} P_n^1(y) \tag{98}$$

Solution

$$A_1 = (-2\alpha\beta\xi_\tau + \beta\delta_\tau)(1 + z\tanh z)\,\mathrm{sech}\, z + C_1 \tanh z\,\mathrm{sech}\, z,$$

$$B_1 = z(2\beta^2\xi_\tau + \beta z)\,\mathrm{sech}\, z + C_2\,\mathrm{sech}\, z.$$

9 Conclusion

1. History
2. Soliton Perturbation Theory
3. Quasistationary Perturbations
4. Direct Integration - KdV
5. Eigenfunction Expansions - KdV
6. Comparison of Results
7. Perturbed NLS

For the damped KdV: Same results.

Care needs to be taken in obtaining solutions to homogeneous problem.

Furthermore, one can compare the quasistationary results to the standard non-quasistationary results. This is given in [3] as

$$u_1(z,t) = \frac{i\gamma}{3\eta} P \int_{-\infty}^{\infty} \frac{1 - e^{ik(k^2+4)\eta^3 t}}{k(k^2+4)^2 \sinh(\frac{\pi k}{2})} p(z,k) e^{ikz}\, dk, \tag{99}$$

where

$$p(z,k) = k(k^2+4) + 4i(k^2+2)\tanh z - 8k\tanh^2 z - 8i\tanh^3 z. \tag{100}$$

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