From Solitary Waves to Rogue Waves
Solutions of Nonlinear Evolution Equations

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Outline

- History of Solitary Waves
- Rise of Nonlinear Wave Research
- Rogue waves
- Emergence of Rogue Wave Research

Small one meter high rogue wave in Baltic Sea, 2007 formed from 30 cm waves.
I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14km/h], preserving its original figure some thirty feet [9m] long and a foot to a foot and a half [300-450mm] in height. Its height gradually diminished, and after a chase of one or two miles [23km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation. John Scott Russell
John Scott Russell (1808-1882)

- Engineer
- Edinburgh
- Used 30 tank
- $v^2 = g(h + a)$

http://www.ma.hw.ac.uk/~chris/scott_russell.html
50 Years of Controversy

George Biddle Airy (1801-1892).

Sir George Gabriel Stokes (1819-1903).
1870s Nonlinear Theory/Solution


\[ u(x, t) = 2\eta^2 \text{sech}^2(\eta(x - 4\eta^2 t)). \]
Korteweg-de Vries Equation - 1895

Gustav de Vries (1866-1934)

Diederik Johannes Korteweg (1848-1941)

\[
\frac{\partial \eta}{\partial t} + \frac{3}{2} \sqrt{g} \frac{\partial}{\partial x} \left( \eta^2 + \frac{3}{2} \alpha \frac{\partial \eta}{\partial x} \right) = 0,
\]

where \( \alpha \) is a small but arbitrary constant, which is in close connection with the exact velocity of the uniform motion given
Navier–Stokes Equations
3 – dimensional – unsteady

Coordinates: \((x,y,z)\)  \hspace{1cm} \text{Time: } t  \hspace{1cm} \text{Density: } \rho  \hspace{1cm} \text{Pressure: } p  \hspace{1cm} \text{Reynolds Number: } Re

Velocity Components: \((u,v,w)\)  \hspace{1cm} \text{Stress: } \tau  \hspace{1cm} \text{Heat Flux: } q  \hspace{1cm} \text{Prandtl Number: } Pr

\textbf{Continuity:} \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0

\textbf{X – Momentum:} \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} + \frac{\partial (\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right]

\textbf{Y – Momentum:} \frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2)}{\partial y} + \frac{\partial (\rho vw)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right]

\textbf{Z – Momentum:} \frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho uw)}{\partial x} + \frac{\partial (\rho vw)}{\partial y} + \frac{\partial (\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re} \left[ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right]

\textbf{Total Energy – Et:} \frac{\partial (E_T)}{\partial t} + \frac{\partial (uE_T)}{\partial x} + \frac{\partial (vE_T)}{\partial y} + \frac{\partial (wE_T)}{\partial z} = -\frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z}

+ \frac{1}{Re} \left[ \frac{\partial}{\partial x} \left( u \tau_{xx} + v \tau_{xy} + w \tau_{xz} \right) + \frac{\partial}{\partial y} \left( u \tau_{xy} + v \tau_{yy} + w \tau_{yz} \right) + \frac{\partial}{\partial z} \left( u \tau_{xz} + v \tau_{yz} + w \tau_{zz} \right) \right]

- \frac{1}{Re Pr} \left[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right]
Derivation - Laplace’s Equation

\[ \mathbf{v} = \nabla \phi \Rightarrow \nabla^2 \phi = 0 \]

Incompressible
\[ \nabla \cdot \mathbf{v} = 0 \]

Irrotational
\[ \nabla \times \mathbf{v} = 0 \]
Derivation - Perturbation Theory

Laplace’s Equation
\[ \phi_{xx} + \phi_{zz} = 0 \]

Kinematic BCs
\[ \eta_t + \phi_x \eta_x - \phi_z = 0 \]
\[ \phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + g\eta = 0 \]
Derivation - KdV Equation

Scaling

\[ \tau = \epsilon^{3/2} t \]
\[ \xi = \epsilon^{1/2} (x - t) \]

Expansions

\[ \eta = \epsilon^{1/2} \sum_{n=0}^{\infty} \epsilon^n \eta_n \]
\[ \phi = \epsilon^{1/2} \sum_{n=0}^{\infty} \epsilon^n \phi_n \]

\[ 2\eta_0 \tau + 3\eta_0 \eta_0 \xi + \frac{1}{3} \eta_0 \xi \xi \xi = 0 \]
KdV Traveling Waves - \( u_t + 6uu_x + u_{xxx} = 0 \)

Seek solutions \( u(x, t) = f(\xi) \), where \( \xi = x - ct \). Using \( u_x = f'(\xi) \) and \( u_t = -cf'(\xi) \), we have

\[-cf' + 6ff' + f''' = 0.\]
Seek solutions $u(x, t) = f(\xi)$, where $\xi = x - ct$. Using $u_x = f'(\xi)$ and $u_t = -cf'(\xi)$, we have

$$-cf' + 6ff' + f''' = 0.$$  

$$[-cf + 3f^2 + f'']' = 0.$$
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\[
-cf' + 6ff' + f''' = 0.
\]

\[
[-cf + 3f^2 + f'']' = 0.
\]

\[
-cf + 3f^2 + f'' = A.
\]
KdV Traveling Waves - $u_t + 6uu_x + u_{xxx} = 0$

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$$[-cf + 3f^2 + f'']' = 0.$$  
$$-cf + 3f^2 + f'' = A.$$  
$$-cff' + 3f^2f' + f''f' = Af'.$$
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$$[-cf + 3f^2 + f'']' = 0.$$

$$-cf + 3f^2 + f'' = A.$$

$$-cff' + 3f^2f' + f''f' = Af'.$$

$$-\frac{1}{2}cf^2 + f^3 + \frac{1}{2}f'^2 = Af + B.$$
KdV Traveling Waves - \( u_t + 6uu_x + u_{xxx} = 0 \)

Seek solutions \( u(x, t) = f(\xi) \), where \( \xi = x - ct \). Using \( u_x = f'(\xi) \) and \( u_t = -cf'(\xi) \), we have

\[
- cf' + 6ff' + f''' = 0.
\]

\[
[-cf + 3f^2 + f'']' = 0.
\]

\[- cf + 3f^2 + f'' = A.\]

\[-cff' + 3f^2f' + f''f' = Af'.\]

\[-\frac{1}{2}cf^2 + f^3 + \frac{1}{2}f'^2 = Af + B.\]

\[
\frac{1}{2} f'^2 = Af + B - (-\frac{1}{2}cf^2 + f^3).
\]
Seek solutions \( u(x, t) = f(\xi) \), where \( \xi = x - ct \). Using \( u_x = f'(\xi) \) and \( u_t = -cf'(\xi) \), we have

\[-cf' + 6ff' + f'''' = 0.\]

\[\left[ -cf + 3f^2 + f'' \right]' = 0.\]

\[-cf + 3f^2 + f'' = A.\]

\[-cf'f + 3f^2f' + f''f' = Af'.\]

\[-\frac{1}{2}cf^2 + f^3 + \frac{1}{2}f'^2 = Af + B.\]

\[\frac{1}{2}f'^2 = Af + B - \left( -\frac{1}{2}cf^2 + f^3 \right).\]

\[\sqrt{\frac{c}{2}} \frac{df}{d\xi} = \sqrt{Af + B + \frac{1}{2}cf^2 - f^3}.\]
KdV Traveling Waves - \( u_t + 6uu_x + u_{xxx} = 0 \)

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\[
\sqrt{\frac{c}{2}} \frac{df}{d\xi} = \sqrt{Af + B + \frac{1}{2}cf^2 - f^3}.
\]

\[
\xi - \xi_0 = \sqrt{\frac{c}{2}} \int \frac{df}{\sqrt{Af + B + \frac{1}{2}cf^2 - f^3}}.
\]
KdV Traveling Waves - $u_t + 6uu_x + u_{xxx} = 0$

Seek solutions $u(x, t) = f(\xi)$, where $\xi = x - ct$. Using $u_x = f'(\xi)$ and $u_t = -cf'(\xi)$, we have

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For $A = B = 0$,

$$\xi - \xi_0 = \sqrt{\frac{c}{2}} \int \frac{df}{\sqrt{\frac{1}{2}cf^2 - f^3}}.$$
KdV Traveling Waves - $u_t + 6uu_x + u_{xxx} = 0$

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For $A = B = 0$,

$$\xi - \xi_0 = \sqrt{\frac{c}{2}} \int \frac{df}{\sqrt{\frac{1}{2}cf^2 - f^3}}.$$  

$$u(x, t) = 2\eta^2 \text{sech}^2(\eta(x - 4\eta^2 t))$$ where $c = 4\eta^2$.  

Soliton Solution - $u(x, t) = 2\eta^2 \text{sech}^2(\eta(x - 4\eta^2 t))$

Time vs position plot of soliton solution, for the KdV $u_t + 6uu_x + u_{xxx} = 0$. 
Dispersion vs Nonlinearity - $u_t + 6uu_x + u_{xxx} = 0$

Dispersion (left): Waves spread and amplitude diminishes.
Nonlinearity (right): Width decreases and waves steepen. (Herman 1992)
Soliton Experiment

Water tank http://www.youtube.com/watch?v=SknvLa8qEu0&feature=related
Re-enactment

July 12, 1995 Union Canal, Scott Russell Aqueduct,
http://apachepersonal.miun.se/~tomnil/solitoner/solipic.htm
Severn Bore - England

The Age of Computing

HISTORIC PHOTOS

Los Alamos scientists Paul Stern (left) and Nick Metropolis playing chess with the MANIAC computer 1951
Fermi, Pasta, Ulam (FPU) Problem (1953-4)

- Simulate 1D chain of masses linked by nonlinear springs.
- Linear - energy of vibration modes remains in the mode
- Nonlinear - allows energy transfer between modes.
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And Mary Tsingou Menzel! - 2008
FPU Problem

\[ m\dddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1})[1 + \alpha(x_{j+1} - x_{j-1})] \]
\[ \dddot{x}_j = \frac{c^2}{h^2}(x_{j+1} - 2x_j + x_{j-1})[1 + \alpha(x_{j+1} - x_{j-1})], \]

where \( c^2 = \frac{Y}{\rho} \), \( \rho = m/h^3 \).

- Surprising behavior: The energy does not drift toward the equipartition predicted by statistical physics but periodically returns to the original mode.

- Los Alamos 1955 report - beginning of nonlinear physics and the age of computer simulations
From FPU to KdV

Let \( x_{j+1}(t) = u(x + h, t) \), \( x_j(t) = u(x, t) \), \( x_{j-1}(t) = u(x - h, t) \):

Use Taylor expansions:

\[
\frac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = u_{xx} + \frac{h^2}{12} u_{xxxx} + O(h^4) \tag{1}
\]

\[
\alpha(x_{j+1} - x_{j-1}) = 2\alpha hu_x + \frac{\alpha h^3}{3} u_{xxx} + O(h^5), \tag{2}
\]

Then,

\[
\frac{1}{c^2} u_{tt} - u_{xx} \approx 2\alpha hu_x u_{xx} + \frac{h^2}{12} u_{xxxx}
\]

Let \( \xi = x - ct \), \( \tau = (\alpha ch)t \), \( y(\xi, \tau) = u(x, t) \),

\[
0 = y_{\tau} + y_{\xi} y_{\xi\xi} - \delta^2 y_{\xi\xi\xi\xi}
\]

\[
0 = v_{\tau} + v v_{\xi} - \delta^2 v_{\xi\xi\xi}\xi. \tag{3}
\]
The KdV Resurgence 1960s

- Kruskal and Zabusky 1965
  - FPU Problem $\rightarrow$ KdV
  - Coined term “soliton”
- Gardner, Greene, Kruskal, Miura 1967
  - Inverse Scattering Transform
  - NLEE revolution
ZK - Recurrence

Emergence of Solitons

- Initial wave
- Shock formation
- Emerged solitons

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From Solitary Waves to Rogue Waves
Soliton Research Begins

- IST - Nonlinear Fourier Transform

\[
\begin{align*}
  u(x, 0) & \xrightarrow{Transform} A(k, 0) \\
  u_t + 6uu_x + u_{xxx} = 0 & \quad A_t = ik^3 A \\
  u(x, t) & \xleftarrow{Inverse Scattering Transform} A(k, t)
\end{align*}
\]

- Solutions of nonlinear PDEs
  - Lax - 1968 \( L\phi = \lambda \phi, \phi_t = B\phi \implies L_t = [L, B] \).
  - Zakharov and Shabat - 1973
  - Ablowitz, Kaup, Newell, Segur - 1974
  - Research extended to other equations, dimensions
  - Other methods for finding exact solutions, \( N \)-soliton solutions
\[ L\phi = -\phi_{xx} + u(x, t)\phi = \lambda \phi \]
\[ \phi_t = B\phi = u_x \phi + (4\lambda - 2u)\phi_x \]
\[ \Rightarrow \quad u_t = [L, B] = -uu_x - u_{xxx}. \]
Two Soliton Solution of KdV

$$u(x, t) = 12 \frac{3 + 4 \cosh(2\xi + 24t) + \cosh(4\xi)}{[3 \cosh(\xi - 12t) + \cosh(3\xi + 12t)]^2},$$

where $\xi = x - 16t$. 
Solitons Elastic Collisions

http://www.scholarpedia.org/article/Soliton
Cnoidal Waves

\[ \eta(x, t) = \eta_2 + H \text{cn}^2 \left( \frac{K(m)}{\lambda} (x - ct) | m \right) \]

where \( \text{cn}(x|m) \) is Jacobi elliptic function and \( K(m) \) is complete elliptic integral.

US Army bombers flying close to the Panama coast (1933).
Two-Dimensional Waves

KadomtsevPetviashvili equation (1970),

\[(u_t + uu_x + u_{xxx})_x + u_{yy} = 0.\]
Sulu Sea
Atmospheric Solitons
Optical Solitons

- 1973 Hasegawa, Tappert predicted optical solitons in communications.
- 1987, First experimental observation in an optical fiber.
- 1988, Mollenauer, et al. transmitted pulses 4,000 km.
- 1991, Bell Labs - transmitted solitons error-free, 2.5 Gb, >14,000 km.
- 1998, Georges, et al. - data transmission of 1 Tb/s ($10^{12}$ bits of information per sec).

Figure 10. Fiber-optic communications could be enhanced by encoding information in solitons. Investigators at AT&T Bell Laboratories have been experimenting with soliton propagation since they first transmitted one through an optical fiber in 1980. Here the oscilloscope trace of a series of solitons is shown before (yellow trace) and after (blue trace) traveling 10,000 kilometers in an optical fiber. The pulses show little tendency to dispense. The transmission rate was five billion bits per second, which would be equivalent to about 100,000 digitized voice channels. (Photograph courtesy of AT&T Bell Laboratories.)
The nonlinear Schrödinger equation:

\[ i \frac{\partial \psi}{\partial t} = \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \psi |\psi|^2. \]

For water waves, \( \eta(x, t) = a(x, t) \cos(kx - \omega t - \theta) \), where \( \psi = ae^{i\theta} \), or \( a = |\psi| \).

\[ \psi(x, t) = 2\beta e^{2\alpha(x+4\alpha t-4(\alpha^2+\beta^2))} \text{sech}(2\beta(x + 4\alpha t + \delta)) \]

Modulational wave solution of NLS.
Peregrine Soliton

- Howell Peregrine (1938-2007)
- 1983 Peregrine predicted spatio-temporal evolution of an NLS soliton
- 20 years later used as prototypical example of rogue waves in water and in optics.
Merchant ship laboring in heavy seas as a huge wave looms astern. Huge waves are common near the 100-fathom line in the Bay of Biscay. Published in Fall 1993 issue of Mariner’s Weather Log. http://en.wikipedia.org/wiki/File:Wea00800,1.jpg
Ship Discoverer gets pounded by monster wave in the Bering Sea.
Charleston, SC - 1991

Rogue wave off of Charleston, South Carolina

Reported collisions with rogue waves:

http://hal.archives-ouvertes.fr/docs/00/00/03/52/PDF/Rogue_wave_V1.pdf
Draupner Wave  Jan 1, 1995

- Oil platform in the central North Sea
- Minor damage
- Read by a laser sensor.
- During wave heights of 12 m (39ft),
  - Freak wave - max height of 25.6 m (84ft)
  - (peak elevation was 18.5 m (61ft)).

Oil platform and time series.
Rational Solutions

NLS Equation,

\[ i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \psi |\psi|^2 = 0. \]

Peregrine solution,

\[ \psi(x, t) = \left( 1 - \frac{4(1 + 2it)}{1 + 4x^2 + 4t^2} \right) e^{it} \]
Rogue Waves 2010

Rogue waves captured
Re-creating monster swells in a tank helps explain their origin

By Devin Powell
June 18th, 2011; Vol.179 #13 (p. 12)

Freak waves that swallow ships whole have been re-created in a tank of water. Though these tiny terrors are only centimeters high, a devilishly difficult mathematical equation describing their shape may help to explain the origins of massive rogue waves at sea...

Sailors have long swapped stories about walls of water leaping up in the open ocean — even in calm water — without warning or obvious cause. But for centuries, rogue waves were little more than talk; no one had ever measured one with scientific instruments.

Then on New Year’s Eve of 1995, a laser on an oil rig off Norway’s coast recorded one of these rare events: a wave 26 meters from bottom to top, flanked by deep troughs on either side.

This wave and others measured since look like breather waves, says Amin Chabchoub, a mathematician at the Hamburg University of Technology in Germany. A breather wave is an...
Fig. 3. Rogue wave triplets. Parameters (a) $\gamma = 20$ and $\beta = 40$; (b) $\gamma = 100$ and $\beta = -400$. 
Darboux transformation of the general Hirota equation: multisoliton solutions, breather solutions, and rogue wave solutions

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Abstract
In this paper, we investigate the exact solutions and conservation laws of a general Hirota equation. Firstly, the N-fold Darboux transformation of this equation is proposed. Then by choosing three kinds of seed solutions, the multisoliton solutions, breather solutions, and rogue wave solutions of the general Hirota equation are obtained based on the Darboux transformation. Finally, the conservation laws of this equation are derived by using its linear spectral problem. The results in this paper may be useful in the study of ultrashort optical solitons in optical fibers.

Keywords: Darboux transformation; multisoliton solutions; breather solutions; rogue wave solutions; optical fibers
History of integrable PDEs (KdV, NLS, etc)
- Solitons, Kinks, Breathers, Loop solitons
- Rational solutions - rogue waves

Solution Techniques
- Inverse Scattering
- Lie Symmetries
- New Solution Methods
- Darboux Transformations
- Perturbation Theory

Rogue waves exist! - 1995 Draupner data

Active area of rogue wave research
- Analytical - new methods of solution generation
- Numerical - robustness of solutions
- Experimental [optics, hydrodynamics, plasmas]
Other Nonlinear Mathematics Topics

- Mandelbrot and Julia sets.
- Iterated function systems.
- Discrete maps and chaos.
- Cardiac dynamics.
- Nonlinear ODE systems.
- Rogue waves.


Bibliography II


