Are Solitary Waves Color Blind to Noise?

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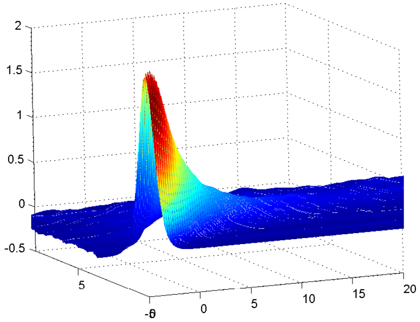

Outline of Talk

What Are Solitary Waves?

Definition

The propagation of non-dispersive energy bundles through discrete and continuous media.

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Example - Burgers' Equation

$$u_t + \alpha u u_x + \beta u_{xx} = 0$$

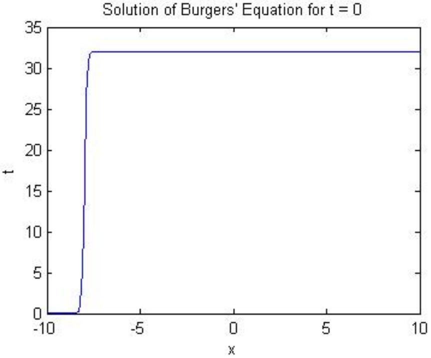
Let $u(x,t) = f(x - ct)$. Then,

$$(-c + \alpha f)f' + \beta f'' = 0.$$

This yields a solution of the form

$$u(x,t) = \frac{c}{\alpha}\left[1 + \beta \tanh \frac{c}{2}(x - ct)\right].$$

Burgers' Equation Traveling Wave



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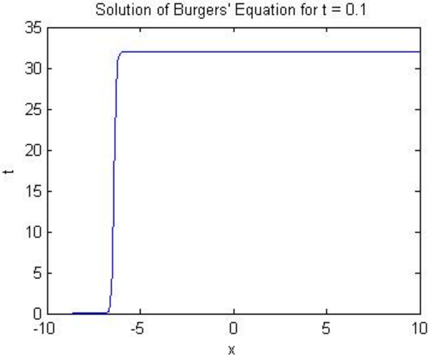

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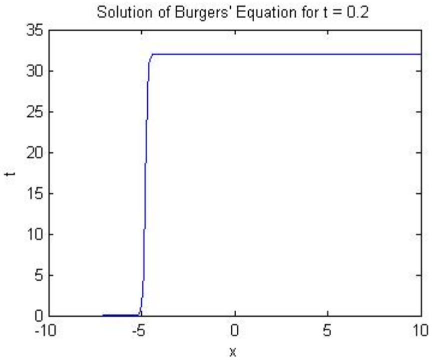

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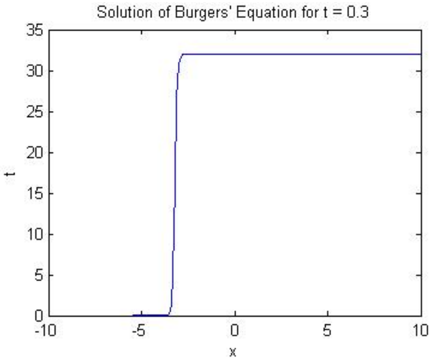

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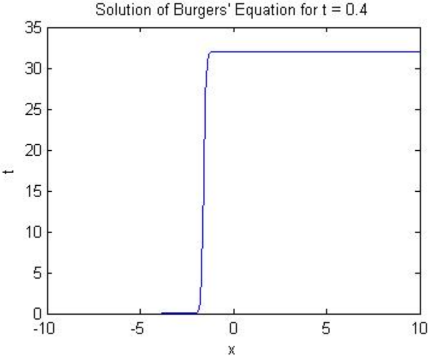

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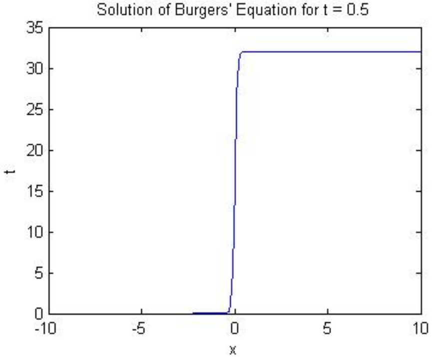

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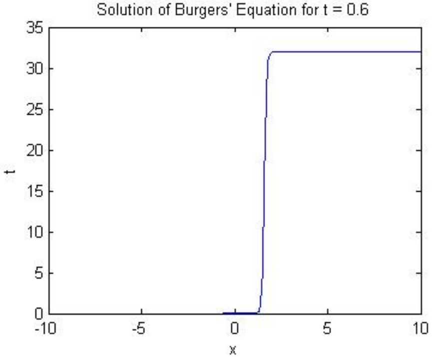

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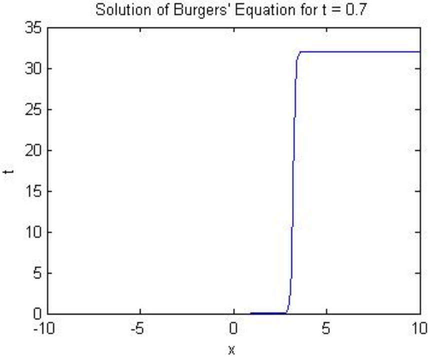

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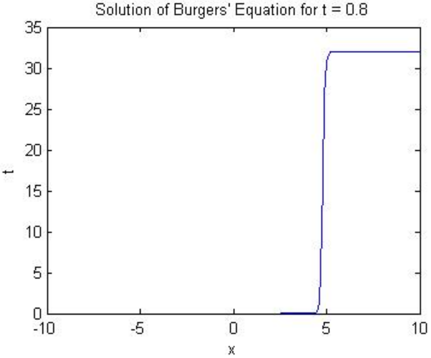

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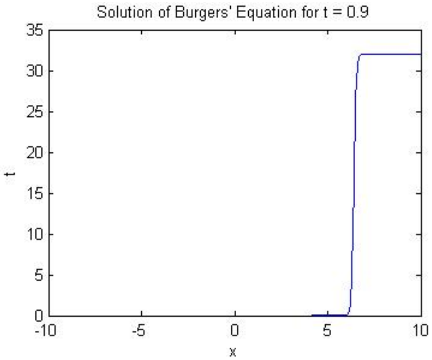

What Are Solitons?

History

- 1834 John Scott Russell's *Great Wave of Translation*
- 50 years of Controversy with Airy, Stokes, et al.
- 1870's Boussinesq and Rayleigh verification
- 1895 Korteweg and deVries derived PDE
- 1965 Zabusky & Kruskal revived KdV Equation in study of Fermi-Pasta-Ulam Problem

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Soliton

Traveling wave solutions satisfying

1. They are of permanent form;
2. They are localised within a region;
3. They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.

The KdV Equation

The PDE

The Korteweg-deVries Equation takes several forms.

$$u_t + \alpha u u_x + \beta u_{xxx} = 0.$$

For example,

$$u_t + 6 u u_x + u_{xxx} = 0.$$

The nonlinear term can balance the dispersive term.

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One - Soliton Solution

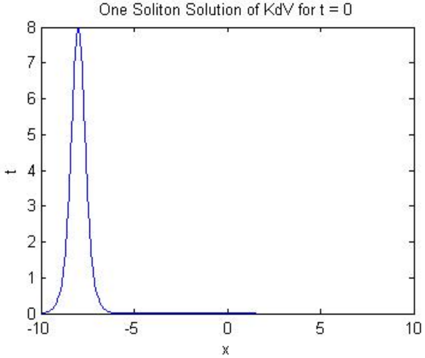
Let $u(x, t) = f(x - ct)$. Then,

$$(-c + 6f)f' + f''' = 0.$$

This yields a solution of the form

$$u(x, t) = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t), \quad c = 4\eta^2.$$

Evolution of One Soliton Solution



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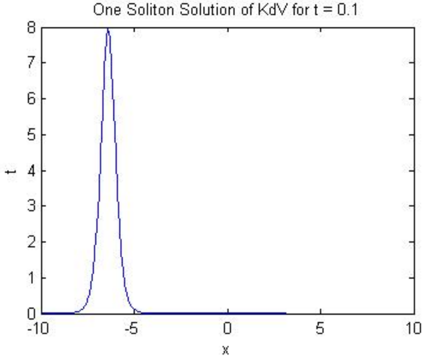

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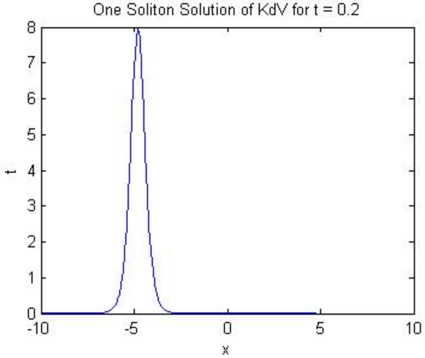

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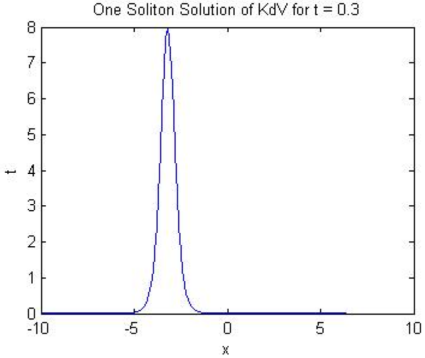

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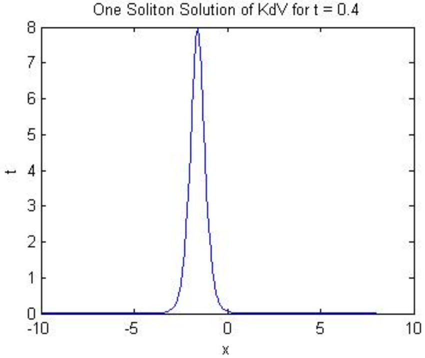

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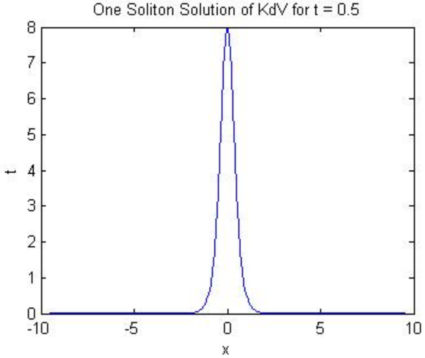

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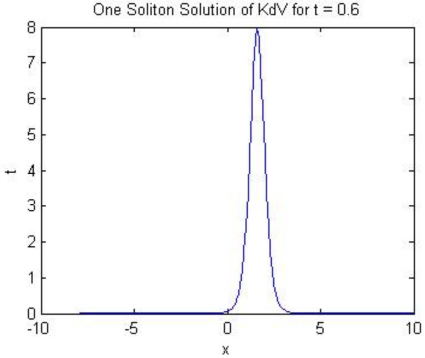

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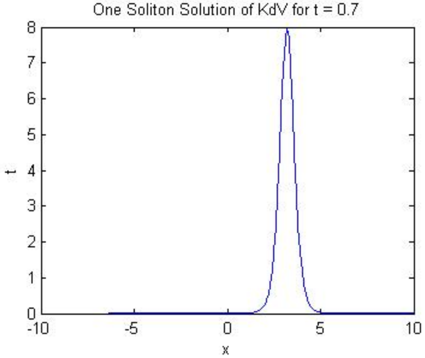

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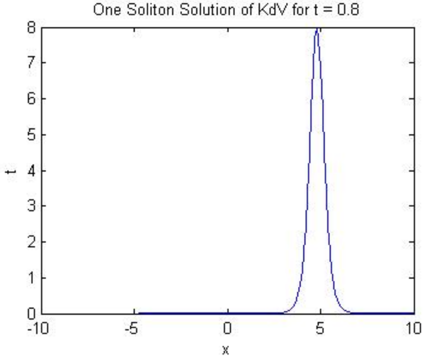

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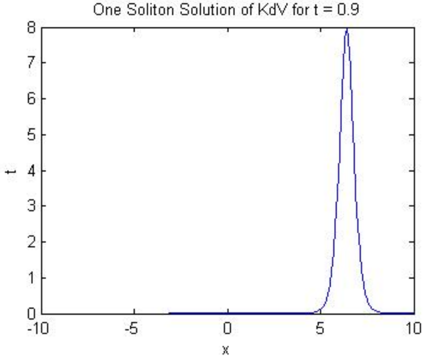


The Two Soliton Solution of the KdV Equation

Form of the Solution

When two solitons collide, they interact elastically. The exact solution for the two soliton equation is given by

$$u(x,t) = \frac{2(p^2 - q^2)(p^2 + q^2 \operatorname{sech}^2 \chi(x,t) \sinh^2 \theta(x,t))}{(p \cosh \theta(x,t) - q \tanh \chi(x,t) \sinh \theta(x,t))^2} \tag{1}$$

where the phases are

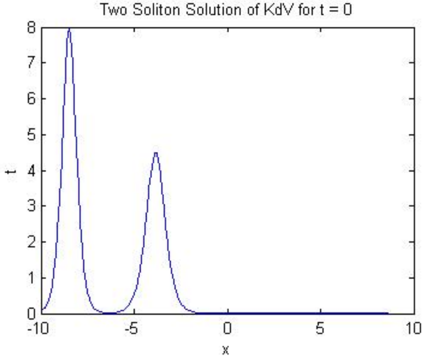
$$\theta(x,t) = px - 4p^3(t - t_0) \tag{2}$$

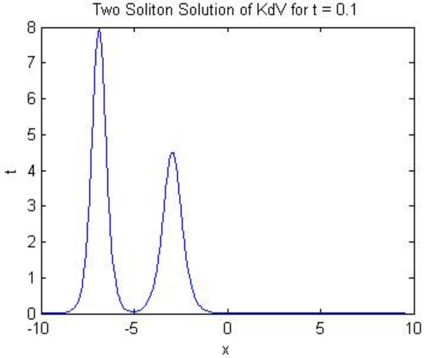
and

$$\chi(x,t) = qx - 4q^3(t - t_0). \tag{3}$$

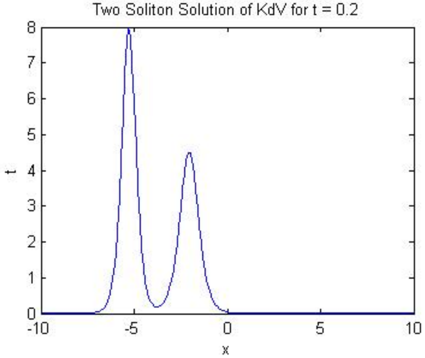
In our simulation we take $p = 2$, $q = 1.5$ and $t_0 = 0.5$.

Animation of Two Solitons Colliding




Two Soliton Solution of KdV for t = 0.1
8
7
6
5
4
3
2
1
0
t
-10
-5
0
5
10
x

Animation of Two Solitons Colliding



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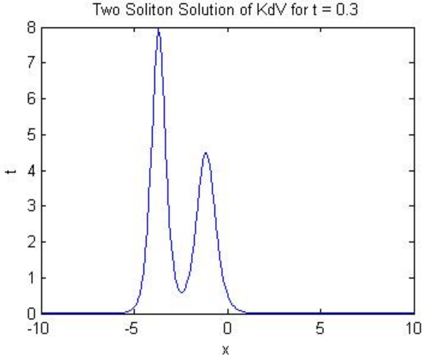

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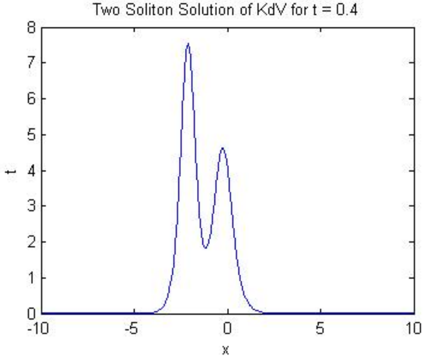

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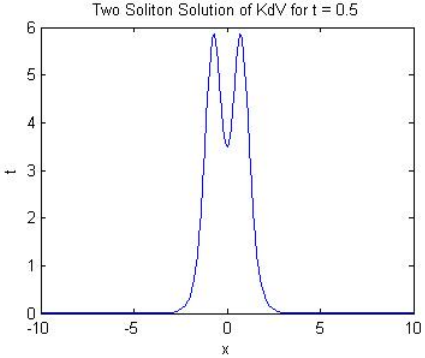

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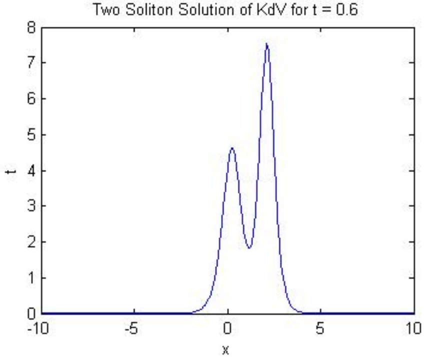

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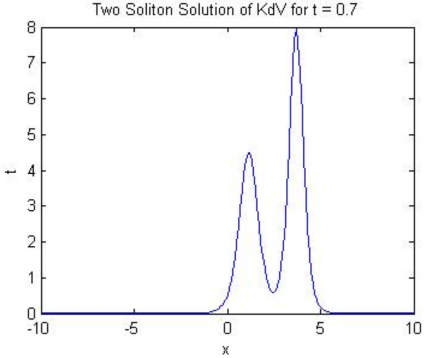

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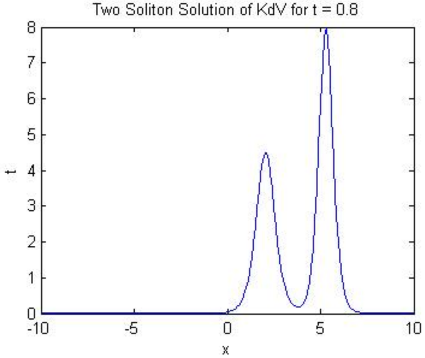

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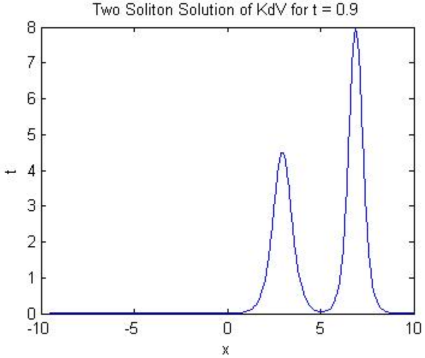
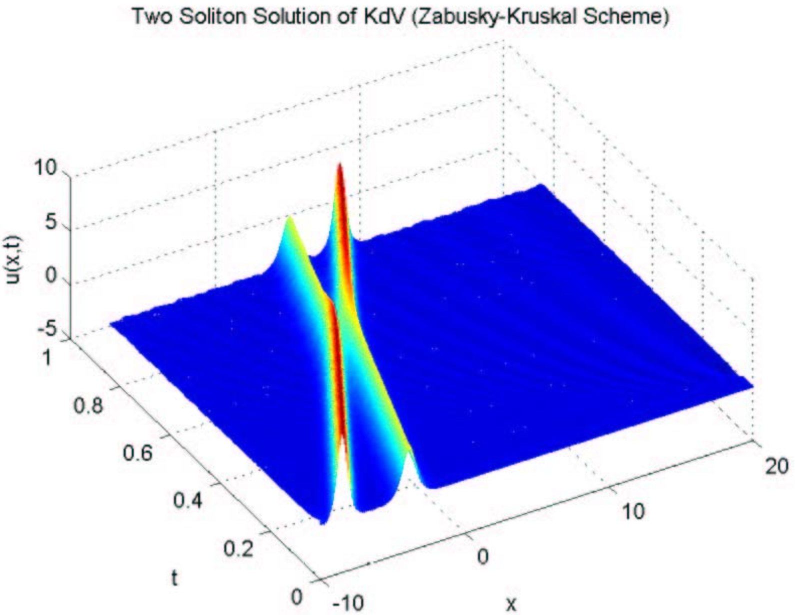



Two Soliton Solution of KdV (Zabusky-Kruskal Scheme)
u(x,t)
10
5
0
-5
1
0.8
0.6
0.4
0.2
0
t
-10
0
10
20
x

Special Cases

The KdV-Burgers Equation is given by

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0.$$

Traveling wave solutions are given for

$$u(x,t) = \frac{2k}{\alpha}\left[1 + \tanh k(x - 2kt)\right] \quad (4)$$

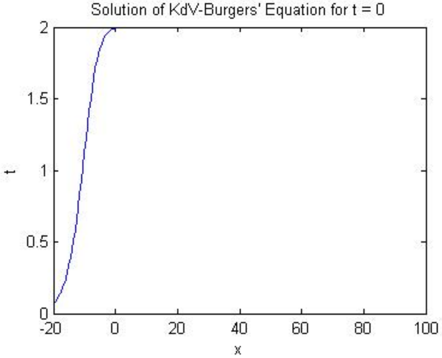
$$u(x,t) = \frac{12sk^2}{\alpha}\operatorname{sech}^2 k(x - 4sk^2 t) \quad (5)$$

$$u(x,t) = A\operatorname{sech}^2\eta(x - vt) + 2A\left[1 + \tanh\eta(x - vt)\right], \quad (6)$$

where

$$A = \frac{3\beta^2}{25\alpha s}, \quad v = \frac{6\beta^2}{5s}, \quad \eta = \frac{\beta}{10s}.$$

KdV-Burgers Traveling Wave Solution



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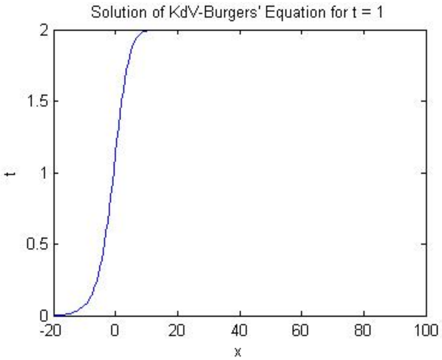

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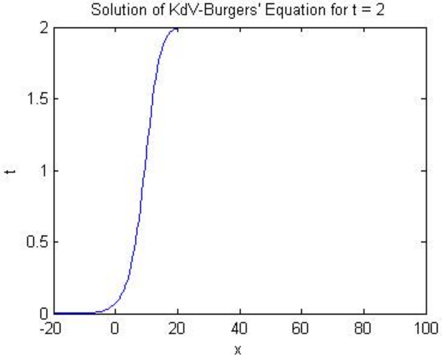

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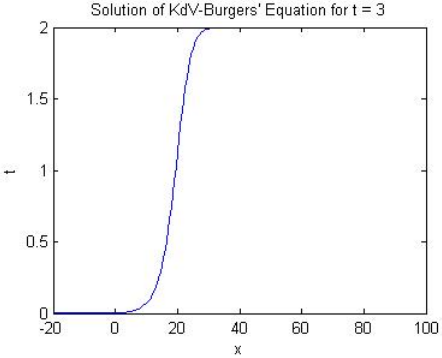

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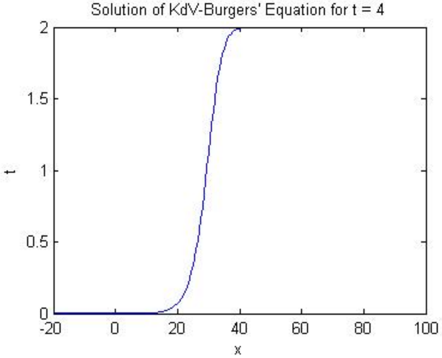

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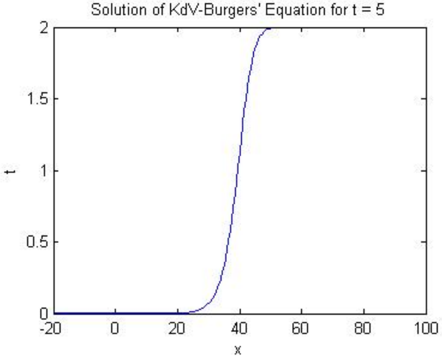

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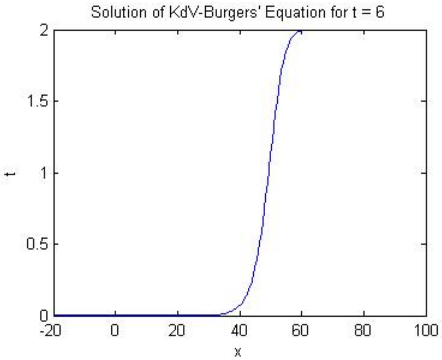

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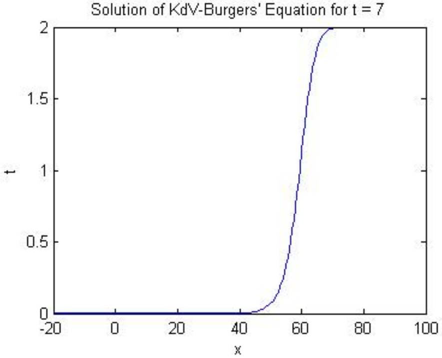

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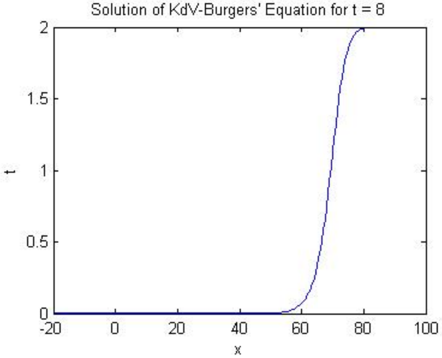

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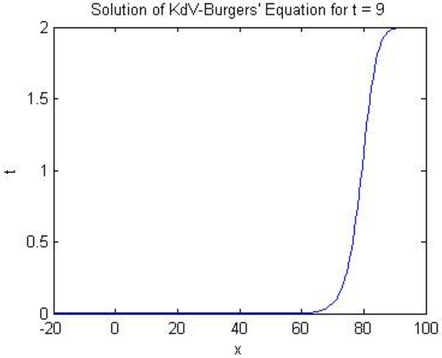

What is Colored Noise?

Noise Types

White - equal energy/cycle - constant frequency spectrum

Pink - $1/f$-noise - flat in log space - decreases 3 dB per octave

Brown - Decrease of 6 dB per octave

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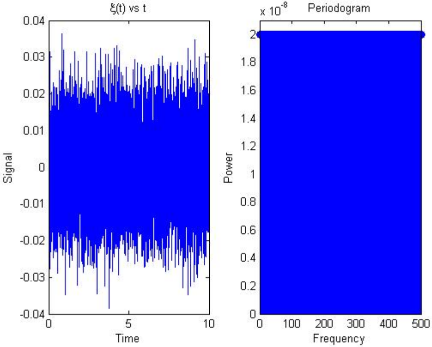

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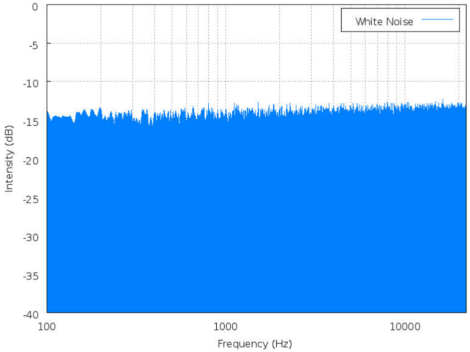

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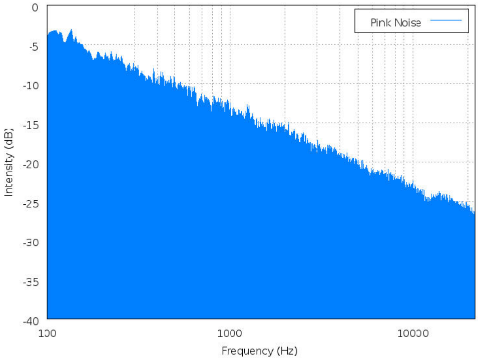

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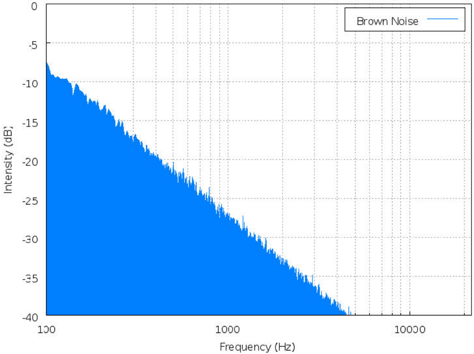

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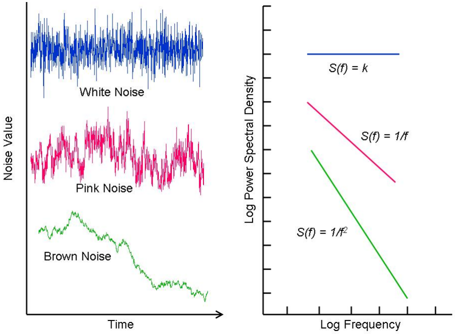

What is White Noise?

White Noise

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$$< \eta(t)\eta(s) >= \delta(t-s).$$

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Brownian Motion

$$\frac{dW}{dt} = \eta(t) \quad \text{or} \quad dW = \eta(t)dt.$$

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Stochastic ODE

$$\frac{dx}{dt} = \epsilon\eta(t)$$

$$dx = \epsilon dW.$$

$$x_i - x_{i-1} = \epsilon(W(t_i) - W(t_{i-1})) \equiv \epsilon dW_i.$$

Real Systems Are Not White

$$\frac{dx}{dt} = \alpha x + \gamma + \mu x \xi(t),$$

where

$$< \xi(t)\xi(s) >= Ae^{-|t-s|/\tau}.$$

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Ornstein-Uhlenbeck Process

$$\frac{d\xi}{dt} = -\frac{1}{\tau}\xi + \frac{\epsilon}{\tau}\eta(t)$$

for

$$< \xi(t) >= 0, \quad < \xi(t)\xi(s) >= \frac{\epsilon^2}{2\tau} e^{-|t-s|/\tau}.$$

Time-dependent SkDV

$$u_t + 6uu_x + u_{xxx} = \zeta(t), \tag{7}$$

$\zeta(t)$ is Gaussian white noise: zero mean and ($< * >= E[*]$)

$$< \zeta(t)\zeta(t') >= 2\epsilon\delta(t - t'). \tag{8}$$

Exact Solution of Stochastic KdV - Wadati - 1983

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Galilean transformation

$$\begin{aligned} u(x,t) &= U(X,T) + W(T), \quad X = x + m(t), \quad T = t, \\ m(t) &= -6\int_0^t W(t')dt', \qquad W(t) = \int_0^t \zeta(t')\,dt', \end{aligned} \tag{9}$$

transforms the stochastic KdV:

$$U_T + 6UU_X + U_{XXX} = 0, \tag{10}$$

The One Soliton Solution Under Noise

The One Soliton Solution of the KdV

$$U(X,T) = 2\eta^2 \operatorname{sech}^2(\eta(X - 4\eta^2 T - X_0)) \tag{11}$$

Leads directly to exact sKdV solution:

$$u(x,t) = 2\eta^2 \operatorname{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right) + W(t). \tag{12}$$

http://people.uncw.edu/hermanr/Research/SKdV/soliton_movie.htm

The One Soliton Solution of the KdV

$$U(X,T) = 2\eta^2 \operatorname{sech}^2(\eta(X - 4\eta^2 T - X_0)) \tag{11}$$

Leads directly to exact sKdV solution:

$$u(x,t) = 2\eta^2 \operatorname{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right) + W(t). \tag{12}$$

http://people.uncw.edu/hermanr/Research/SKdV/soliton_movie.htm

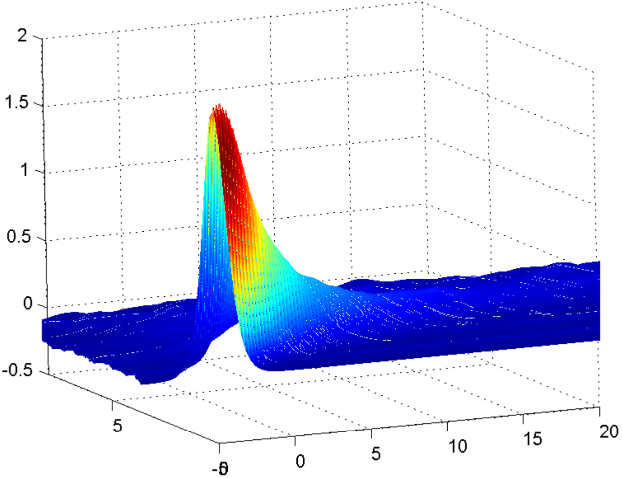
Exact Statistical Average

$$\begin{aligned} < u(x,t) > &= \frac{4\eta^2}{\pi}\int_{-\infty}^{\infty} \frac{\pi k}{\sinh \pi k} e^{iak - bk^2}\,dk. \\ &= \frac{\eta^2}{\sqrt{\pi b}}\int_{-\infty}^{\infty} e^{-(a-s)^2/4b} \operatorname{sech}^2 \frac{s}{2}\,ds, \end{aligned} \tag{13}$$

where

$$a = 2\eta(x - x_0 - 2\eta^2 t),$$

Averaged Soliton with Noise



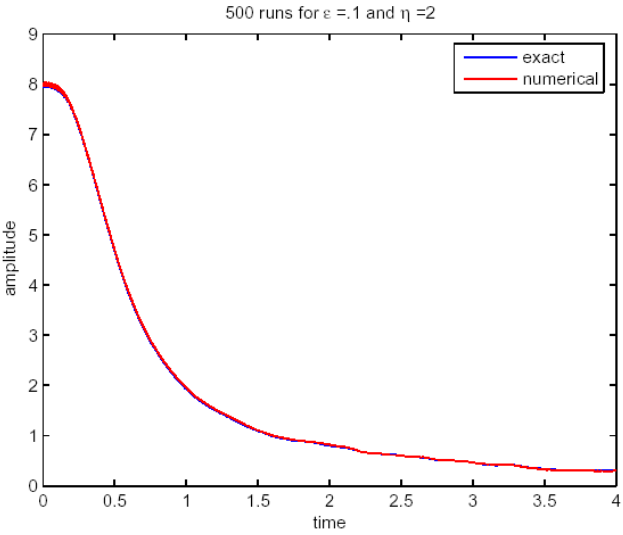


Figure: 500 runs for $x \in [-10, 90]$ with $N = 1000$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - White

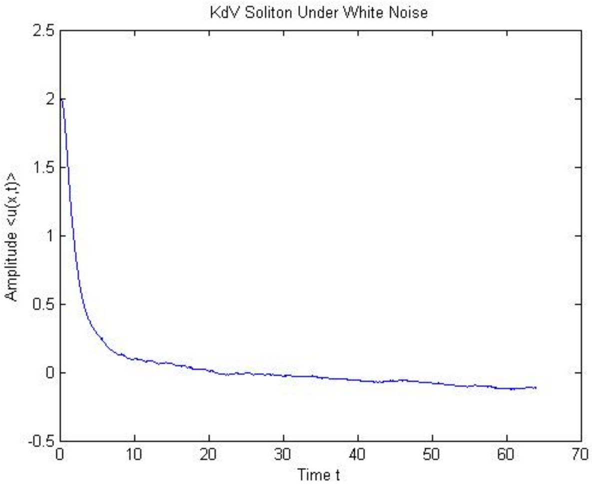


Figure: 500 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Pink Noise

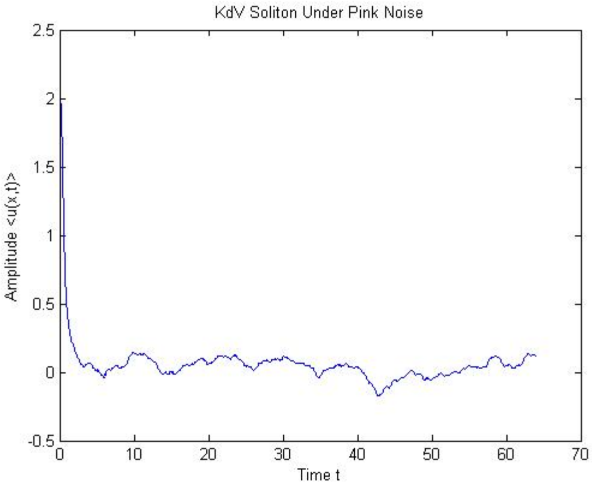


Figure: 1000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Pink Noise

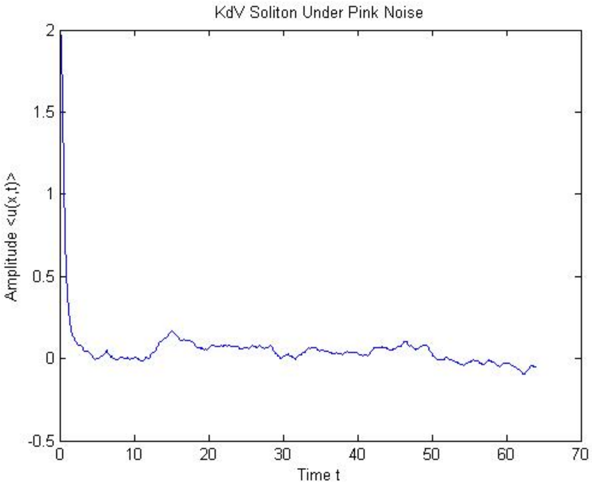


Figure: 2000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Brown Noise

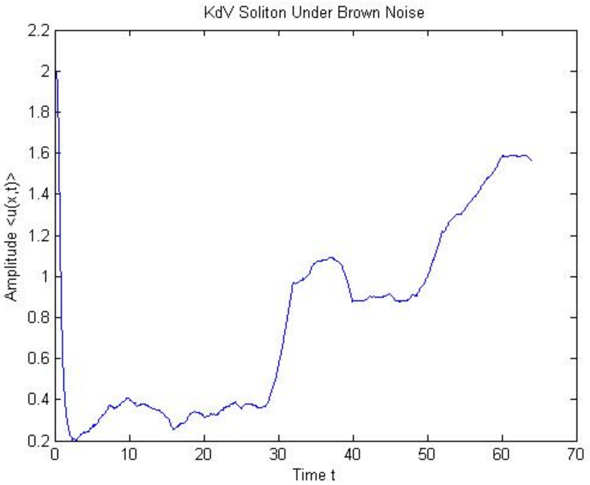


Figure: 1000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Brown Noise

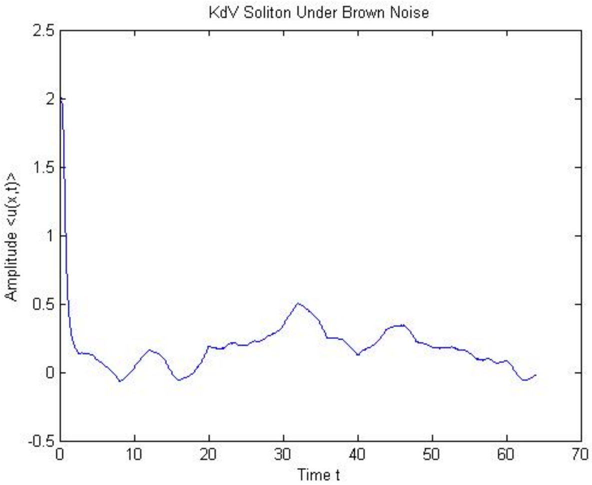


Figure: 2000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Brown Noise

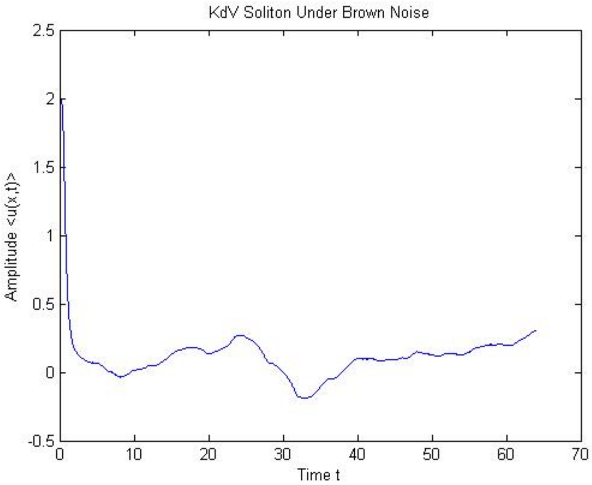


Figure: 3000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Outline of Talk

1. Solitary Waves and Solitons
2. White Noise and Colored Noise?
3. Quantifying Colored Noise
4. The Exact Solution of Stochastic KdV Equation
5. Numerical Results *to date*
6. Summary

So, What is the answer?

What's the question?

Are Solitary Waves Color Blind to Noise?

So, What is the answer?

What's the question?

Are Solitary Waves Color Blind to Noise?

Answer

Most likely - NO
Thank you.

R. L. Herman, "Solitary Waves," *American Scientist*, vol. 80, July-August 1992, 350-361.

R. L. Herman and C. J. Knickerbocker, "Numerically Induced Phase Shift in the KdV Soliton," *Journal of Computational Physics*, vol. 104, no. 1, January 1993, 50-55.

Desmond J. Higham, "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations," *SIAM Review*, vol. 43, no. 3, 2001, 525-546.

A. C. Vliegenthart, "On Finite-Difference Methods for the Korteweg-de Vries Equation," *Journal of Engineering Mathematics*, vol. 5, no. 2, April 1971, 137-155.

Miki Wadati, "Stochastic Korteweg-de Vries Equation," *Journal of the Physical Society of Japan*, vol. 52, no. 8, August 1983, 2642-2648.

Miki Wadati and Yasuhiro Akutsu, "Stochastic Korteweg-de Vries Equation with and without Damping," *Journal of the Physical Society of Japan*, vol. 53, no. 10, October 1984, 3342-3350.