Propagation of Solitons Under Colored Noise

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Outline of Talk

1. Solitary Waves and Solitons
2. White Noise and Colored Noise
3. Exact Solutions of the Stochastic KdV
4. Numerical Results *to date*
5. Summary

Solitary Waves and Solitons

Solitary Waves

The propagation of non-dispersive energy bundles through discrete and continuous media.

Example - Burgers' Equation

$$u_t + \alpha u u_x + \beta u_{xx} = 0, \quad u(x,t) = \frac{c}{\alpha}\left[1 + \beta \tanh \frac{c}{2}(x - ct)\right].$$

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Solitons

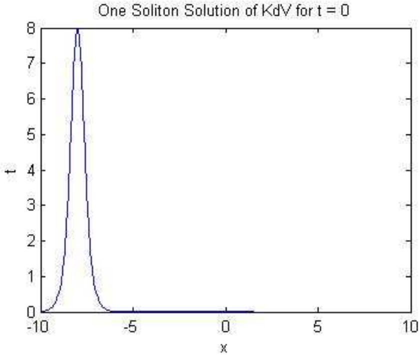
Traveling wave solutions satisfying

1. They are of permanent form;
2. They are localised within a region;
3. They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.

Example - KdV Equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad u(x,t) = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t), \quad c = 4\eta^2.$$

Evolution of One Soliton Solution



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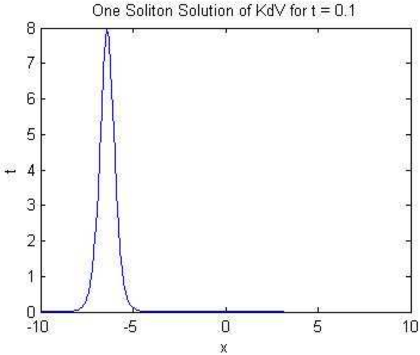

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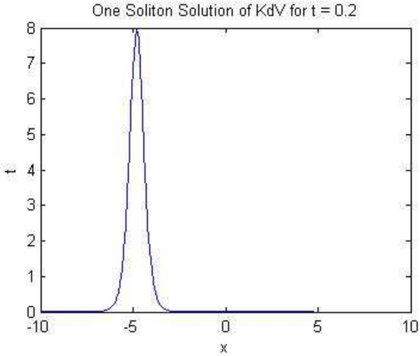

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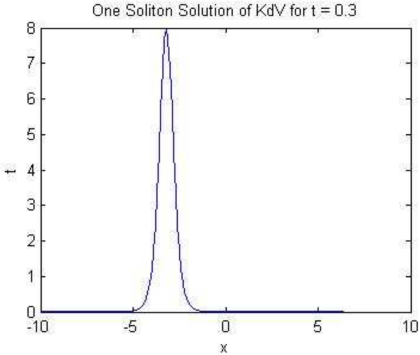

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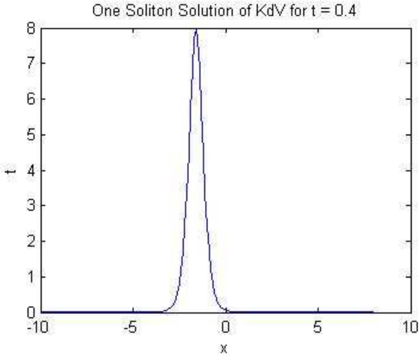

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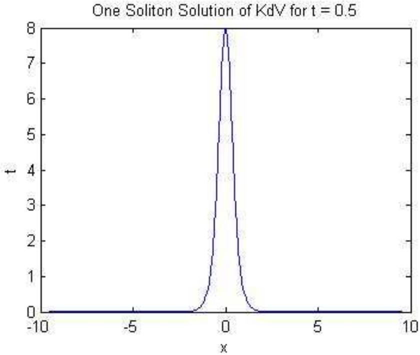

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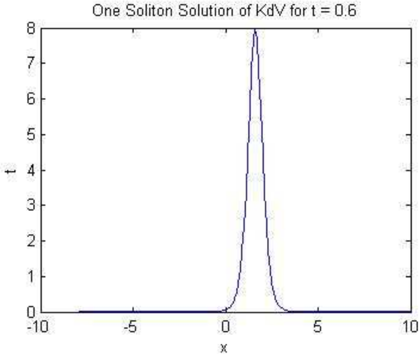

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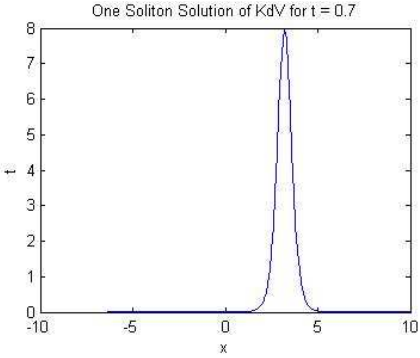

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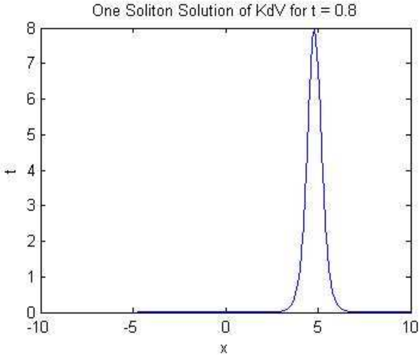

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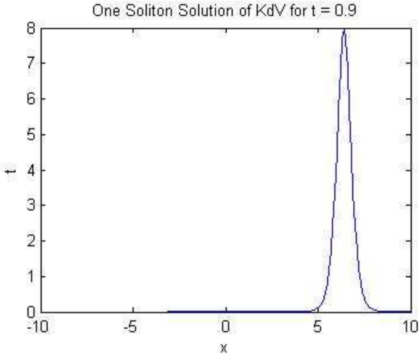


Form of the Solution

When two solitons collide, they interact elastically. The exact solution for the two soliton equation is given by

$$u(x,t) = \frac{2(p^2 - q^2)(p^2 + q^2 \operatorname{sech}^2 \chi(x,t) \sinh^2 \theta(x,t))}{(p \cosh \theta(x,t) - q \tanh \chi(x,t) \sinh \theta(x,t))^2} \tag{1}$$

where the phases are

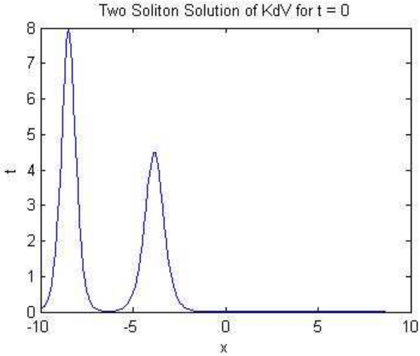
$$\theta(x,t) = px - 4p^3(t - t_0) \tag{2}$$

and

$$\chi(x,t) = qx - 4q^3(t - t_0). \tag{3}$$

In our simulation we take $p = 2$, $q = 1.5$ and $t_0 = 0.5$.

Animation of Two Solitons Colliding



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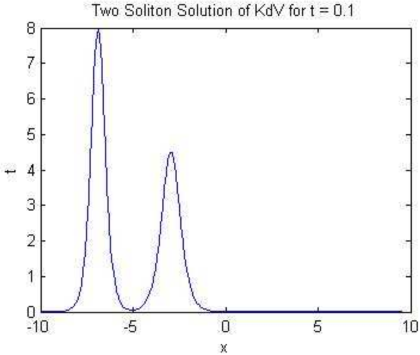

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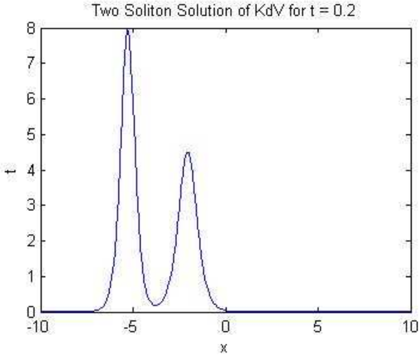

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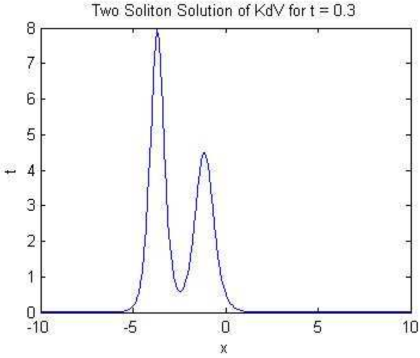

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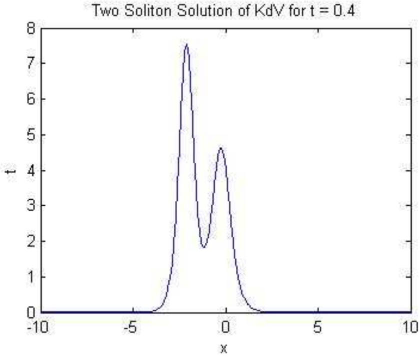

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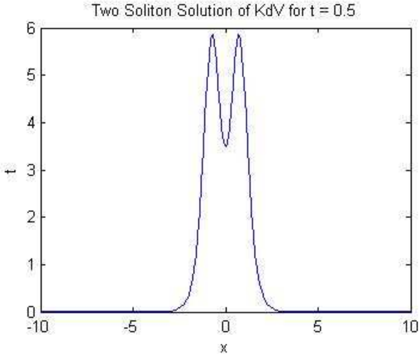

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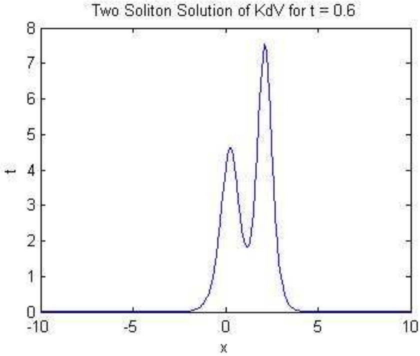

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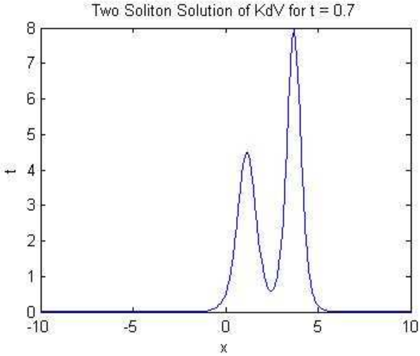

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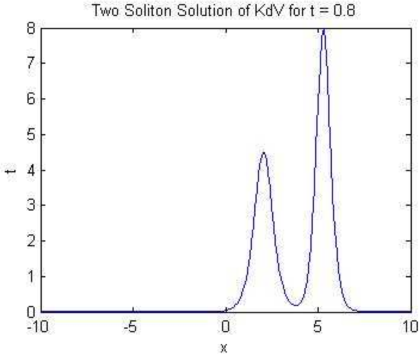

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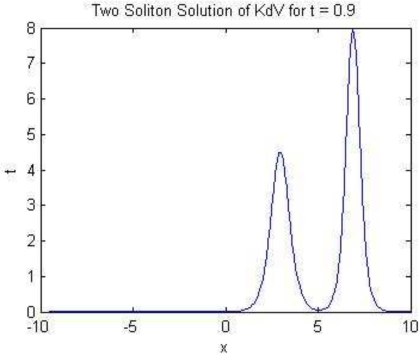
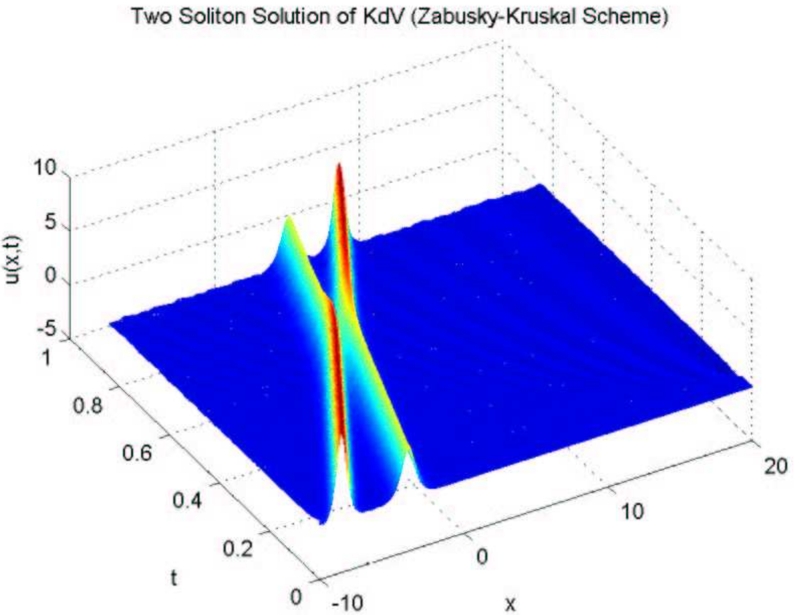


The Two Soliton Solution of the KdV Equation



KdV-Burgers Equation

Special Cases

The KdV-Burgers Equation is given by

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0.$$

Traveling wave solutions are given for

$$u(x,t) = \frac{2k}{\alpha}\left[1 + \tanh k(x - 2kt)\right] \quad (4)$$

$$u(x,t) = \frac{12sk^2}{\alpha}\operatorname{sech}^2 k(x - 4sk^2 t) \quad (5)$$

$$u(x,t) = A\operatorname{sech}^2 \eta(x - vt) + 2A\left[1 + \tanh \eta(x - vt)\right], \quad (6)$$

where

$$A = \frac{3\beta^2}{25\alpha s}, \quad v = \frac{6\beta^2}{5s}, \quad \eta = \frac{\beta}{10s}.$$

White Noise and Colored Noise

Noise Types

White - equal energy/cycle - constant frequency spectrum

Pink - $1/f$-noise - flat in log space - decreases 3 dB per octave

Brown - Decrease of 6 dB per octave

Blue - Increase 3 dB per octave

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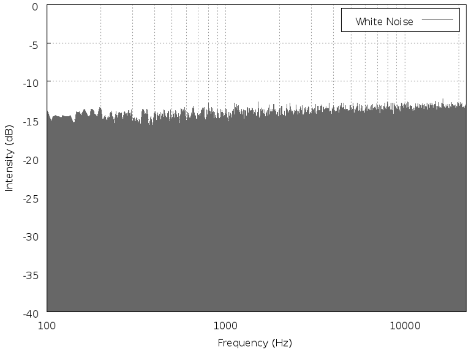

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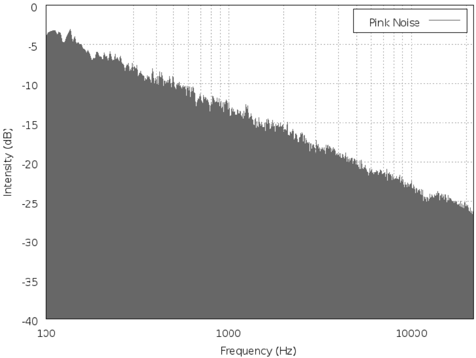

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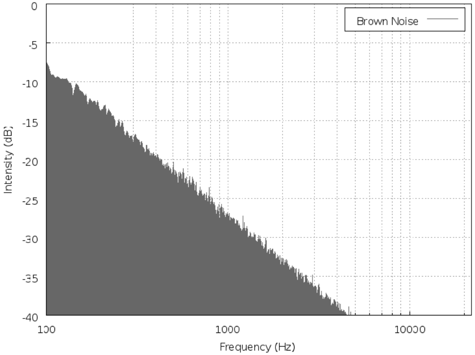

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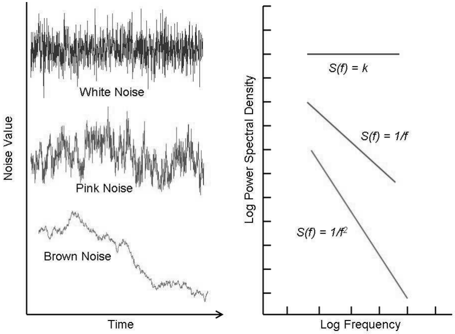

Quantifying White Noise

White Noise

$$< \eta(t) >= 0.$$

$$< \eta(t)\eta(s) >= \delta(t-s).$$

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Stochastic ODE

$$\frac{dx}{dt} = \epsilon\eta(t)$$

$$dx = \epsilon dW.$$

$$x_i - x_{i-1} = \epsilon(W(t_i) - W(t_{i-1})) \equiv \epsilon dW_i.$$

Quantifying Colored Noise

Exponentially Correlated (Real) Noise - Ornstein-Uhlenbeck Process

$$\frac{d\xi}{dt} = -\frac{1}{\tau}\xi + \frac{\epsilon}{\tau}\eta(t)$$

for

$$< \xi(t) >= 0, \quad < \xi(t)\xi(s) >= \frac{\epsilon^2}{2\tau} e^{-|t-s|/\tau}.$$

Gaussian and stationary if

$$< \xi(0) >= 0, \quad < \xi^2(0) >= \frac{\epsilon^2}{2\tau}$$

$$p(\xi_0) = \sqrt{\frac{\tau}{\epsilon\pi} e^{-\xi_0^2\tau/\epsilon}}$$

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$1/f^\beta$-noise, Kaulakys, et al., Physica A, 365 (2006) 217-221

$$\frac{dx}{dt} = \Gamma x^{2\sigma-1} + x^\sigma \eta(t), \quad \beta = 2 - \frac{2\Gamma + 1}{2\sigma - 2}$$

Exact Solution of the Stochastic KdV Equation

Problem

$$u_t + 6uu_x + u_{xxx} = \zeta(t), \tag{7}$$

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Galilean transformation - Wadati 1983

$$\begin{aligned} u(x,t) &= U(X,T) + W(T), \quad X = x + m(t), \quad T = t, \\ m(t) &= -6\int_0^t W(t')dt', \qquad W(t) = \int_0^t \zeta(t')\,dt', \end{aligned} \qquad (8)$$

transforms the stochastic KdV: $U_T + 6UU_X + U_{XXX} = 0.$

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transforms the stochastic KdV: $U_T + 6UU_X + U_{XXX} = 0.$

The One Soliton Solution of the KdV

$$U(X,T) = 2\eta^2 \operatorname{sech}^2(\eta(X - 4\eta^2 T - X_0)) \tag{9}$$

$$u(x,t) = 2\eta^2 \operatorname{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt'\right)\right) + W(t). \tag{10}$$

Ensemble (Statistical) Average

General

$$
\begin{aligned}
< u(x,t) > &= \frac{4\eta^2}{\pi} \int_{-\infty}^{\infty} \frac{\pi k}{\sinh \pi k} e^{iak - bk^2}\, dk. \\
&= \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{-(a-s)^2/4b} \operatorname{sech}^2 \frac{s}{2}\, ds, \qquad (11)
\end{aligned}
$$

where $a = 2\eta(x - x_0 - 2\eta^2 t)$,

$$
b = 2\eta^2 < m^2(t) >, \quad m(t) = -6 \int_0^t W(t')dt'.
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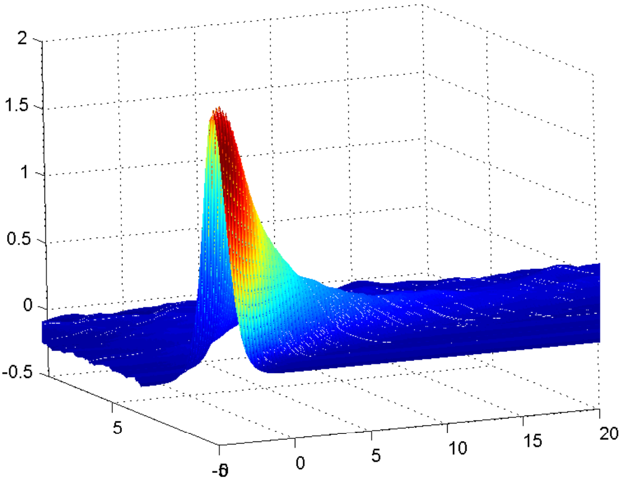
Results

White Noise (Wadati 1983): $b = 48\eta^2 \epsilon t^3$
Exponential noise (Orlowski and Sobczyk 1989):

$$b = 36\eta^2\epsilon^2\tau^3 \left(\frac{2}{3}(\frac{t}{\tau})^3 - (\frac{t}{\tau})^2 - 2(1 - \frac{t}{\tau})e^{-t/\tau} + 2\right)$$

Averaged Soliton with Noise



Stochastic KdV: $< u(x,t) >_{max}$ vs. t

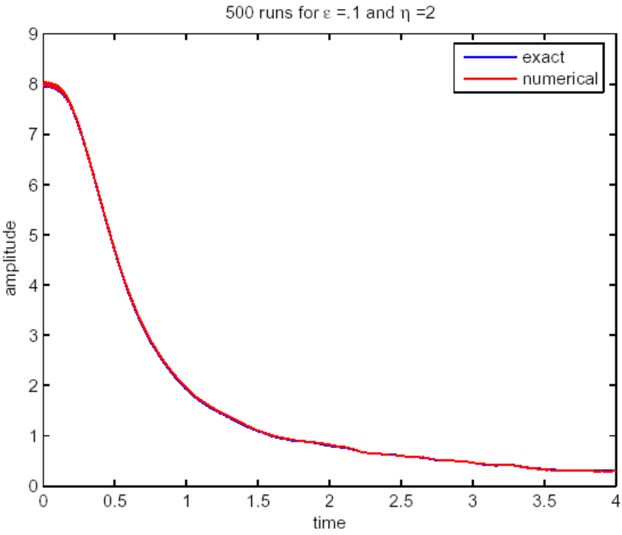


Figure: 500 runs for $x \in [-10, 90]$ with $N = 1000$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV: $\log(e^{2\gamma t/3} < u(x,t) >_{max})$ *vs.* $\log t$

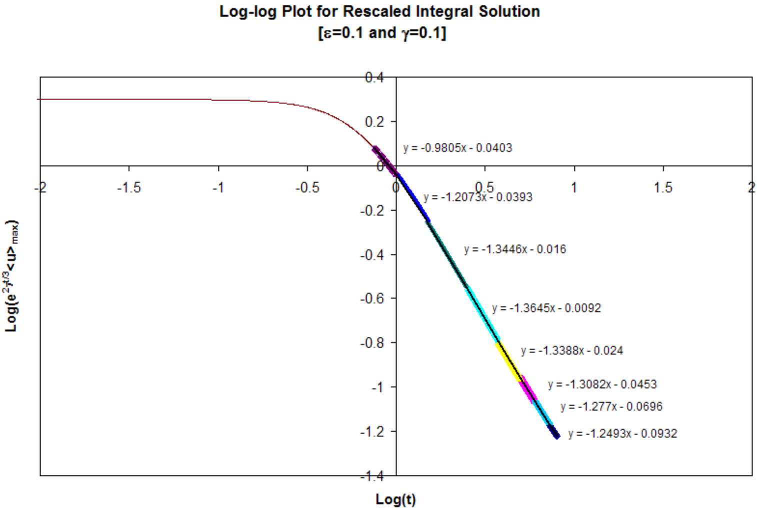


Figure: One Soliton Amplitude with Damping and Noise.

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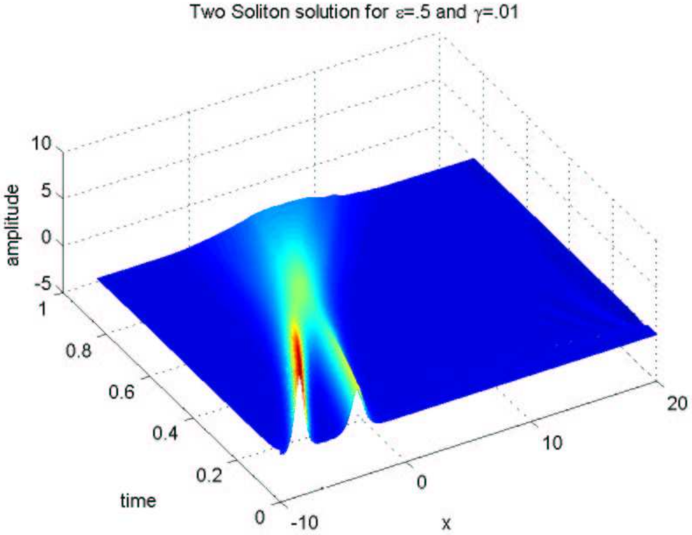


Figure: Two Soliton Solution with Damping and Noise.

What About Colored Noise vs White?

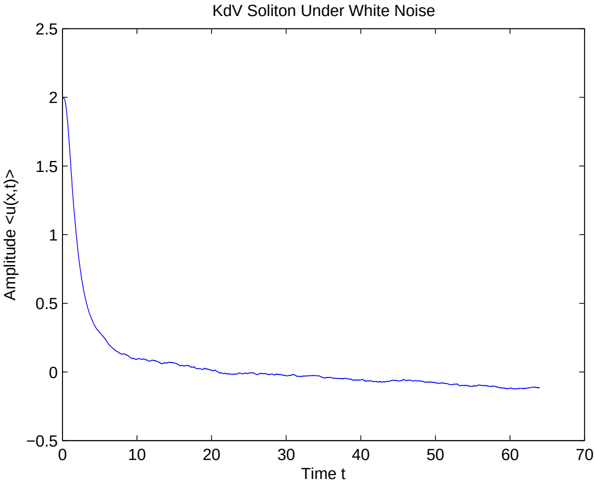
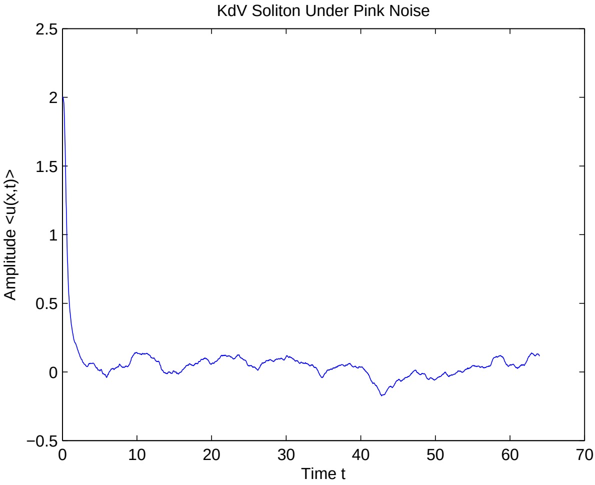


Figure: 500 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Pink Noise



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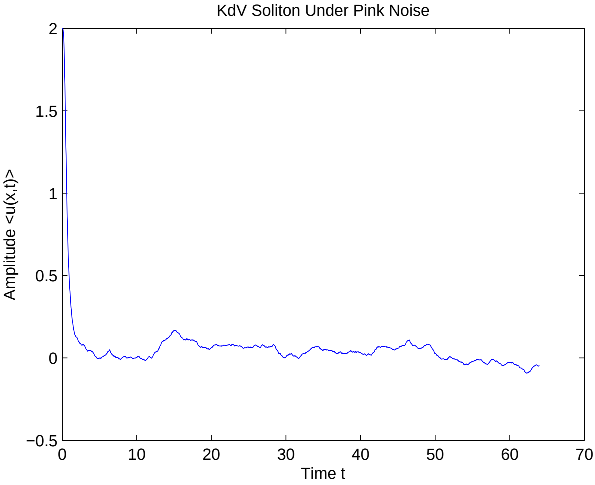
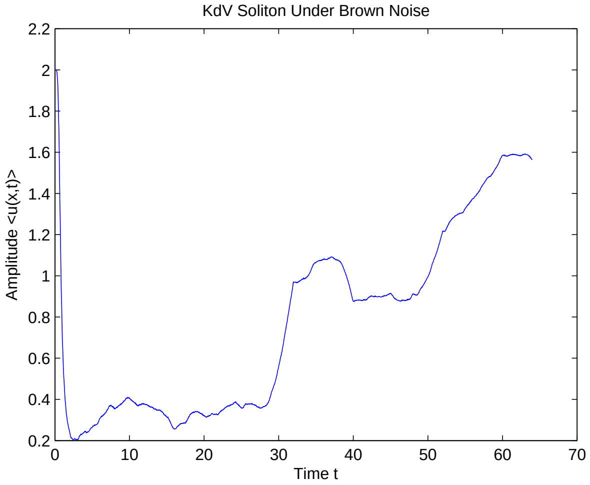


Figure: 2000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

Stochastic KdV Amplitudes - Brown Noise



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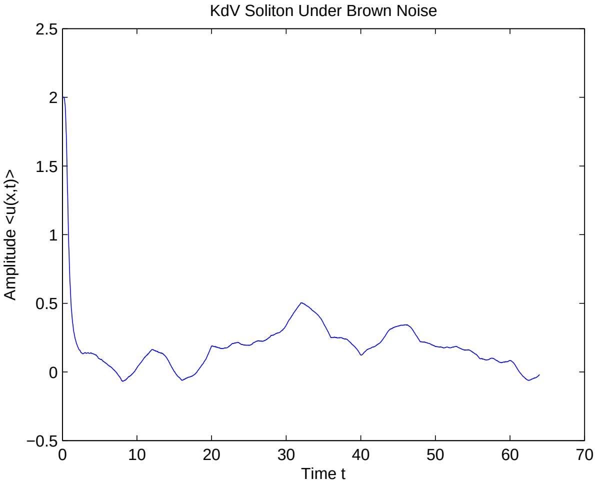

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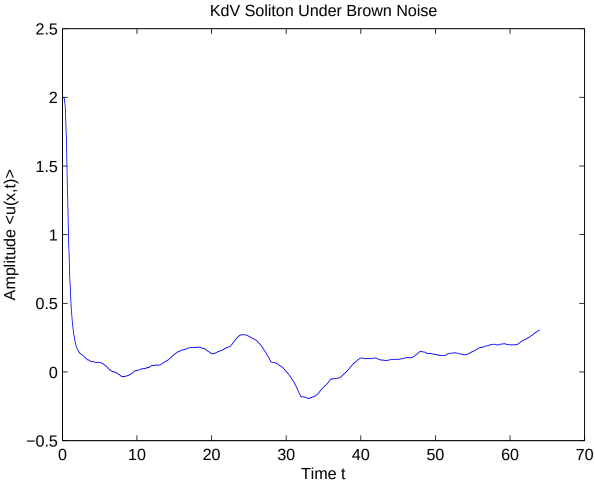

Figure: 3000 runs for $x \in [-10, 90]$ with $N = 500$, $\epsilon = .1$, and $\eta = 2$.

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Sample References

- Herman R L, 1990 "The Stochastic, Damped KdV Equation" *J. Phys. A* **23** 1063-1084.
- Herman R L and Rose A, 2008 "Numerical Realizations of Solutions of the Stochastic KdV Equation", to appear in *Mathematics and Computers in Simulation*.
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