MAT 415-515

Introduction to Complex Variables

$x^2+1=0 \Rightarrow x^2=-1$

or $x=\pm\sqrt{-1} \equiv \pm i$

Complex Numbers: $a+bi$, $a,b\in\mathbb{R}$

Algebraic Structure

Consider set of ordered pairs $(x,y)$, $x,y\in\mathbb{R}$

Let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$

Equality $z_1=z_2$ iff $x_1=x_2$ and $y_1=y_2$

Equivalence Relation:

1. $z_1=z_1$, $\forall z_1$
2. $z_1=z_2 \Rightarrow z_2=z_1$ $\forall z_1,z_2$
3. $z_1=z_2$ and $z_2=z_3 \Rightarrow z_1=z_3$, $\forall z_1,z_2,z_3$

Addition $z_1+z_2=z_3$ iff $x_1+x_2=x_3$ and $y_1+y_2=y_3$

1. $z_1+z_2=z_2+z_1$ (commutative)
2. $z_1+(z_2+z_3)=(z_1+z_2)+z_3$
3. $\exists 0 \ni \forall z,\ z+0=z$
4. $\forall z\ \exists -z \ni z+(-z)=0$

#3 $z+(\alpha,\beta)=z$

$(x,y)+(\alpha,\beta)=(x,y)$

$x+\alpha=x$ and $y+\beta=y$

or $\alpha=0$, $\beta=0$.

$0=(0,0)$

Uniqueness of 0

Assume $0, 0'$ are additive identities

Then

$$\left.\begin{aligned} 0'+0 &= 0' && \text{by 3} \\ 0'+0 &= 0+0' && \text{by 1} \\ &= 0 && \text{by 3} \end{aligned}\right\} \Rightarrow 0'=0$$

Subtraction $z_1-z_2=z_1+(-z_2)$

multiplication $z_1 z_2 = z_3$ iff

$$x_3 = x_1x_2 - y_1y_2$$
$$y_3 = x_1y_2 + x_2y_1$$

Properties

1. $z_1z_2 = z_2z_1, \quad \forall z_1, z_2$
2. $z_1(z_2z_3) = (z_1z_2)z_3, \ \forall z_1, z_2, z_3$
3. $\exists$ identity $1 \ni \forall z, \ 1z = z$
4. $\forall z \neq 0, \exists z^{-1} \ni \ z^{-1}z = 1$

3. $1z = z$

$(\alpha, \beta)z = z$

$\Rightarrow \alpha x - \beta y = x$
$\quad\ \ \alpha y + \beta x = y$ $\quad \Rightarrow \alpha = 1, \beta = 0$

$1 = (1, 0)$

4. Find $z^{-1} = (\xi, \eta)$

$$\left.\begin{array}{l} (\xi, \eta)z = 1 \\ \xi x - \eta y = 1 \\ \xi y + \eta x = 0 \end{array}\right\} \quad \begin{array}{l} \xi = \dfrac{x}{x^2+y^2} \\ \eta = -\dfrac{y}{x^2+y^2} \end{array}$$

Note! $x^2+y^2 \neq 0 \Rightarrow x, y \neq 0$ or $z = 0$

Division: $\dfrac{z_1}{z_2} = z_1 z_2^{-1} = z_3$

$$x_3 = \frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2}$$

$$y_3 = \frac{-x_1y_2 + x_2y_1}{x_2^2+y_2^2}$$

Collection of ordered pairs of reals with equality and the above addition, multiplication $= \mathbb{C}$

– algebraic field over $\mathbb{R}$

$$z_1(z_2+z_3) = z_1z_2 + z_1z_3 \quad \forall z_1, z_2, z_3 \in \mathbb{C}$$

Consider $z = (0, 1)$

$$z^2 = (0^2 - 1^2, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -(1, 0) = -1$$

So $\quad i = (0, 1)$

$\mathbb{R} \subset \mathbb{C}$ since $(x, 0) \in \mathbb{C}, \ x \in \mathbb{R}$

$\mathbb{R} \subset \mathbb{C}$ since $(x,0) \in \mathbb{C}$, $x \in \mathbb{R}$

$\mathbb{R}$ is isomorphic to the set of reals.

– $\mathbb{R}$ is an ordered field.

Real Part of $z = x$ or $\mathrm{Re}(z) = x$

Imaginary part of $z$ $\mathrm{Im}(z) = y$

Define $az = (ax, ay)$ $a \in \mathbb{R}$, $z \in \mathbb{C}$

1. $(a+b)z = az + bz$
2. $a(z_1 + z_2) = az_1 + az_2$
3. $a(bz) = abz$
4. $1z = z$

+ Addition properties $\Rightarrow$ $\mathbb{C}$ is a linear vector space over $\mathbb{R}$

<u>Note</u> $(a,b) = a(1,0) + b(0,1)$

$= a + bi$

Normally $z = x + iy$

Complex Modulus

$$|z| = \sqrt{x^2+y^2} \geq 0$$

$$|z| = 0 \iff z = 0$$

(or $x=0, y=0$)

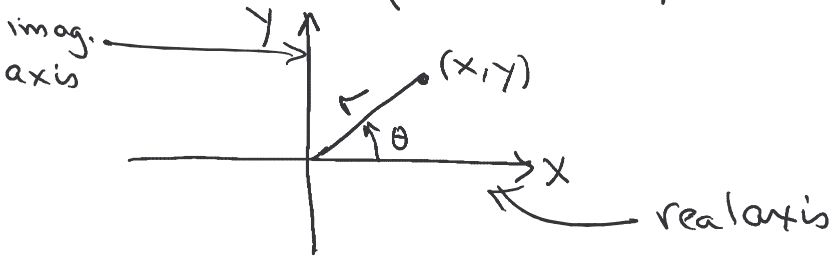
Complex Conjugate, $\bar{z} = x - iy$

Note $z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2 = |z|^2$

$\mathbb{C}$ – linear vector space over $\mathbb{R}$

– set ordered pairs $z = (x, y)$




## Polar Representation

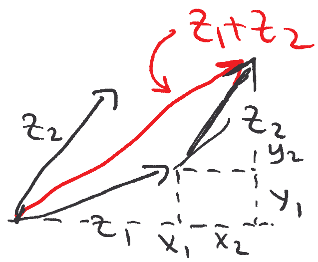
$$z = x + iy$$
$$= r\cos\theta + ir\sin\theta$$

$|z| = r$

$$\theta = \arg z$$
$$= \tan^{-1}\left(\frac{y}{x}\right)$$

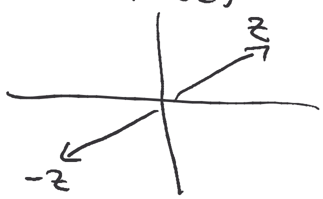
## Geometry

$z_1 + z_2$

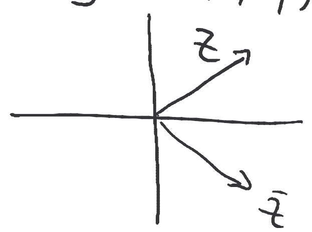



$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

$-z$




$\bar{z} = x - iy \quad = (x, -y)$




$$\arg z = -\arg \bar{z}$$
$$\arg(-z) = \pi + \arg z$$

**Define** $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

So $z = r(\cos\theta + i\sin\theta)$ – before

$$\Rightarrow \boxed{z = re^{i\theta}}$$

**Ex** $z = 1 + i$

$$r = \sqrt{2}$$
$$\theta = \tan^{-1}(1)$$
$$= \pi/4$$

$r = \sqrt{2}$

$\theta = \tan^{-1}(1)$

$= \pi/4$

$z = \sqrt{2}\, e^{i\pi/4}$

Ex $t = \sqrt{1+i} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{1/2}$

$t^2 = 1+i$

$t^2 - (1+i) = 0 \Rightarrow 2$ solutions

Need $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$

$e^{4\pi i} = 1$

$\Rightarrow e^{2k\pi i} = 1$, $k$-integer

Now, $\sqrt{1+i} = \left(\sqrt{2}\, e^{i\pi/4} \cdot 1\right)^{1/2}$

$= \left(\sqrt{2}\, e^{i\pi/4} e^{2k\pi i}\right)^{1/2}$

$= \sqrt[4]{2}\, e^{i\pi/8} e^{\pi k i}$

$= \sqrt[4]{2}\, e^{i\pi/8}, -\sqrt[4]{2}\, e^{i\pi/8}$

$\boxed{e^{i\pi} = -1}$

Other Properties:

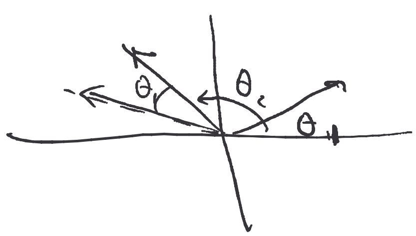
$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$|e^{i\theta}| = \sqrt{e^{i\theta} e^{-i\theta}} = 1$

$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$




$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}, \quad z_2 \neq 0$

Inequalities

## Inequalities

$$\mathrm{Re}(z) \le |z|, \quad \mathrm{Im}(z) \le |z| \qquad (z = x+iy)$$

Pf/ $\mathrm{Re}(z) = x \le \sqrt{x^2} \le \sqrt{x^2+y^2} = |z|$

Triangle Inequality

$$|z_1+z_2| \le |z_1| + |z_2|$$

Pf/
$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$
$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= z_1\bar{z}_1 + \underbrace{z_1\bar{z}_2 + z_2\bar{z}_1} + z_2\bar{z}_2$$

Note: $\overline{z_1\bar{z}_2} = \bar{z}_1\bar{\bar{z}}_2 = z_2\bar{z}_1$

and $z + \bar{z} = (x+iy)+(x-iy) = 2x = 2\,\mathrm{Re}(z)$

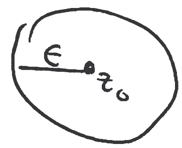
So,
$$|z_1+z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2$$
$$\le |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$$
$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$
$$= (|z_1|+|z_2|)^2 \qquad \underline{\text{QED}}$$

Others
$$|z_1| = |z_1-z_2+z_2| \le |z_1-z_2| + |z_2|$$
$$\Rightarrow \quad |z_1| - |z_2| \le |z_1-z_2|$$

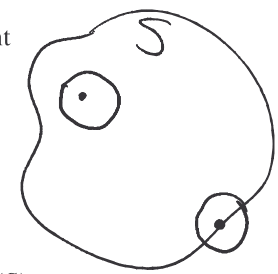
Similarly,
$$|z_2| - |z_1| \le |z_1-z_2|$$
$$\Rightarrow \quad |z_1-z_2| \ge \big|\,|z_1|-|z_2|\,\big|$$

## Point Sets

S




| | |
|---|---|
| Complement | $C(S)=\{z \mid z\notin S\}$ |
| Subset | $S_1\subseteq S_2$ if $z\in S_1 \Rightarrow z\in S_2$ |
| Union | $S_1\cup S_2=\{z \mid z\in S_1 \text{ or } z\in S_2\}$ |
| Intersection | $S_1\cap S_2=\{z \mid z\in S_1 \text{ and } z\in S_2\}$ |
| ε-neighborhood of $z_0$ | $N_\varepsilon=\{z \mid \lvert z-z_0\rvert<\varepsilon\}$ |
| Limit Point of $S$ from $z_0$ | If every ε-neighborhood of $z_0$ contains at least one point different |
| Interior Point of $S$ | If some ε-neighborhood of $z_0$ contains only points in $S$ |
| Open | If $S$ contains only interior points |
| Closed | If $S$ contains all of its limit points or if it has no limit points |
| Boundary Point of $S$ | If every ε-neighborhood of $z_0$ contains both points in $S$ and in $C(S)$. |
| Boundary of $S$ | Collection of boundary points. |
| Closure | $\overline{S}=S\cup S'$, for $S'$ the set of limit points |
| Empty Set | $\varnothing$ |
| Bounded | If there exists $M$ such that $\lvert z\rvert<M, \forall z\in S$. |




$\phi$

### Comments

| | |
|---|---|
| Proper Subset | $S_1\subseteq S_2$, $z\in S_2$ and $z\notin S_1$ |
| Equality | $S_1\subseteq S_2$ and $S_2\subseteq S_1$ implies $S_1=S_2$ |
| Disjoint | $S_1\cap S_2=\varnothing$ |

$z_0$ is a limit point of $S \Rightarrow$ every ε-neighborhood of $z_0$ contains an infinite number of * points.

Finite sets are closed.

The complement of an open set is closed.

There are sets that are both open and closed. There are sets that are neither.

Compact Sets — If every infinite subset has a limit point.
Compact sets are closed and bounded

**Bolzano-Weierstrass Thm** Every bounded infinite set has at least on limit point.

Covering — Collection of sets $\{C_\alpha\}$ such that $\forall \alpha\in S, \exists C_\alpha \ni \alpha\in C_\alpha$

**Heine-Borel Thm** If $S$ is closed and bound and has an open covering, then there is a finite subcovering of $S$.

* $z_0$ is a limit point

Let $\lvert z_0 - z_1\rvert < \epsilon$.

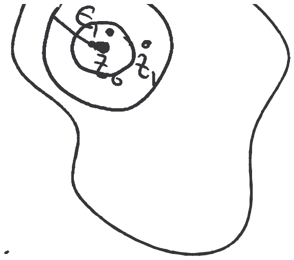
* $z_0$ is a limit point

Let $|z_0 - z_1| < \epsilon_1$

Pick $\epsilon_2 = \frac{1}{2}|z_0 - z_1|$

$\exists\, z_2 \ni |z_0 - z_2| < \epsilon_2$

Repeat $\Rightarrow z_1, z_2, z_3, \ldots$



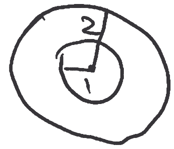
Examples: Open/Closed Sets

Open $\phi, \mathbb{C}, \{z \mid |z|<1\}, \operatorname{Im}(z) > 0$

UHP - Upper half plane

Closed $\mathbb{C}, \phi, \{z \mid |z| \le 1\}, \operatorname{Re}(z) \ge 0$

Neither $\{z \mid 1 < |z| \le 2\}$




## Compact Sets

- if every infinite subset has a limit pt.

$\equiv$ closed and bounded

Pf/ 1) $S$ is finite or $\phi$

There are no infinite subsets $\Rightarrow$ no lim pts

trivial - closed, bounded $\Rightarrow$ compact

2) $S$ is infinite & compact.

Let $z_0$ be a limit point of $S$

$\exists\, z_1 \in N_{\epsilon_1}(z_0)$ or $|z_1 - z_0| < \epsilon_1$

$\neq z_0$

Let $\frac{1}{2}|z_1 - z_0| = \epsilon_2$ Then $\exists\, z_2 \in N_{\epsilon_2}(z_0)$

with $z_2 \neq z_0, z_1$

repeating this process $\Rightarrow$

$z_1, z_2, z_3, \ldots$ an infinite sequence of distinct $z$'s $\subseteq S$ with limit point $z_0$

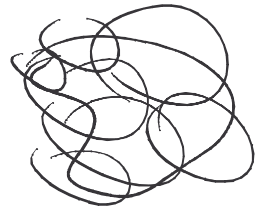
$S$ compact $\Rightarrow z_0 \in S$

So all limit points $\in S \Rightarrow S$ closed.

Is $S$ bounded? Assume $S$ is not bounded

Then $\exists z_1 \in S \ni. |z_1| > 1$
and $\exists z_2 \in S \ni. |z_2| > 2$
.... $\exists z_n \in S \ni. |z_n| > n$.
for any integer $n$
$z_1, z_2, z_3 \ldots$ w/o limit pt. $\otimes$

Coverings



# Stereographic Projection

$\mathbb{C}$ - not compact

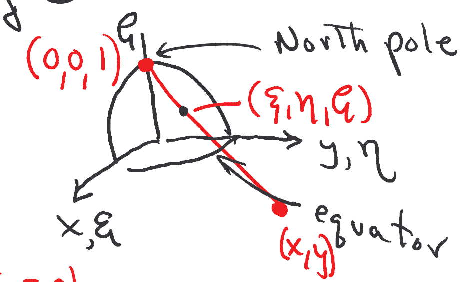
Consider $z=1,2,3,\ldots$ has no limit pt.

Want to compactify $\mathbb{C}$

Riemann Sphere

$z = x+iy$

$\xi^2+\eta^2+\zeta^2=1$



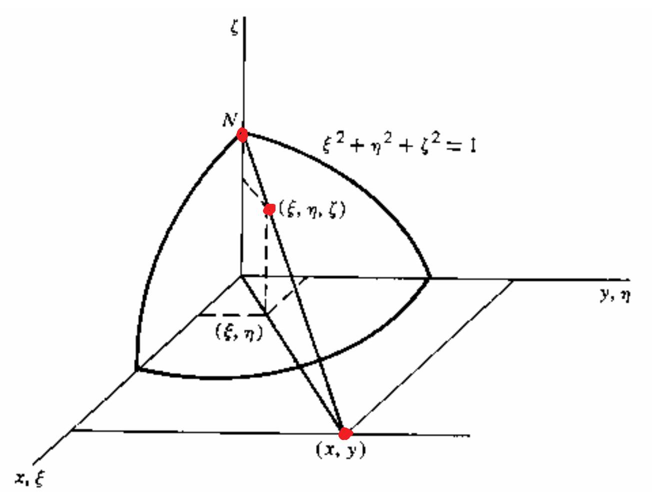

$z$ is mapped to the sphere

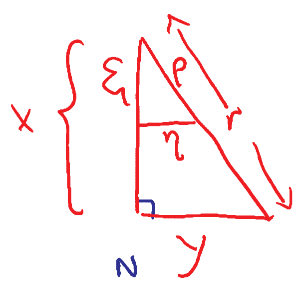
$(\xi,\eta,\zeta)$ - image of $z$

is almost 1-1 Not North Pole.

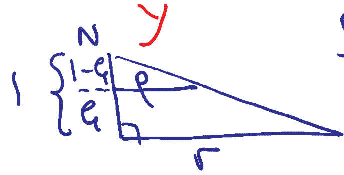
To include the North Pole - add the point at $\infty$. $\Rightarrow$ extended complex plane

Transformations




$$\frac{\xi}{x}=\frac{\eta}{y}=\frac{\rho}{r}$$




$$\frac{\rho}{r}=\frac{1-\zeta}{1}$$

$$\frac{\xi}{x}=\frac{\eta}{y}=\frac{\rho}{r}=\frac{1-\zeta}{1},$$

where $r=\sqrt{x^2+y^2}$ and $\rho=\sqrt{\xi^2+\eta^2}$. From this, we obtain

$$x=\frac{\xi}{1-\zeta},\ y=\frac{\eta}{1-\zeta},\ \xi=\frac{2x}{r^2+1},\ \eta=\frac{2y}{r^2+1},\ \zeta=\frac{r^2-1}{r^2+1}.$$

$\xi^2+\eta^2+\zeta^2=1$

$\rho^2+\zeta^2=1$

$$x^2+y^2=\frac{\xi^2+\eta^2}{(1-\zeta)^2}$$

$$r^2=\frac{\rho^2}{(1-\zeta)^2}=\frac{1-\zeta^2}{(1-\zeta)^2}$$

$$1+r^2=1+\frac{1-\zeta^2}{(1-\zeta)^2}=\frac{(1-\zeta)^2+1-\zeta^2}{(1-\zeta)^2}$$

$2(1-\zeta)$

$\alpha z\bar{z}+\hat{\beta}z+\hat{\gamma}\bar{z}+\delta=0$

$$1+r^2 = 1+\frac{1+\zeta}{(1-\zeta)^2} = \frac{(1-\zeta)+1-\zeta}{(1-\zeta)^2}$$
$$= \frac{2(1-\zeta)}{(1-\zeta)^2}$$
$$= \frac{2}{1-\zeta}$$
So
$$1-\zeta = \frac{2}{1+r^2}$$

stereographic projection. For example, *straight lines and circles are mapped into circles.* The general equation of a circle in the complex plane is

$$\alpha x^2 + \alpha y^2 + \beta x + \gamma y + \delta = 0.$$

Under stereographic projection, we have

$$\alpha \frac{\xi^2+\eta^2}{(1-\zeta)^2} + \frac{\beta\xi}{1-\zeta} + \frac{\gamma\eta}{1-\zeta} + \delta = 0,$$
$$\Rightarrow \alpha \frac{1-\zeta^2}{(1-\zeta)^2} + \frac{\beta\xi}{1-\zeta} + \frac{\gamma\eta}{1-\zeta} + \delta = 0,$$
$$\Rightarrow \alpha(1+\zeta) + \beta\xi + \gamma\eta + \delta(1-\zeta) = 0,$$
$$\Rightarrow \beta\xi + \gamma\eta + (\alpha-\delta)\zeta + \alpha + \delta = 0.$$

plane in $\xi,\eta,\zeta$ space

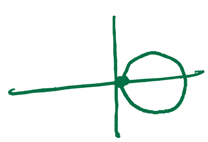
Need the intersection of the plane and the unit sphere = circle or a pt.

Ex $\alpha=0$ $\quad \beta\xi+\gamma\eta-\delta\zeta+\delta=0$ & $\xi^2+\eta^2+\zeta^2=1$

and $\quad \beta x+\gamma y+\delta=0$ — line

Note the circles pass through $N: (\xi,\eta,\zeta)=(0,0,1)$

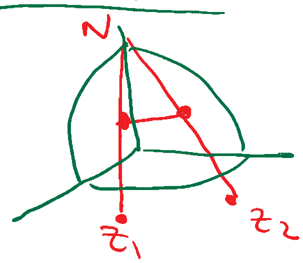
Ex $\delta=0, \alpha\neq 0$



$\alpha x^2+\alpha y^2+\beta x+\gamma y=0$ passes thru origin $(0,0)$

But $(0,0) \longrightarrow$ South Pole $(0,0,-1)$.

Chordal Metric




$$\rho(z_1,z_2) = \sqrt{(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2}$$
$$= \sqrt{2(1-\xi_1\xi_2-\eta_1\eta_2-\zeta_1\zeta_2)}$$
$$= \frac{2\sqrt{r_1^2+r_2^2-2x_1x_2-2y_1y_2}}{\sqrt{1+r_1^2}\sqrt{1+r_2^2}}$$

$$\rho(z_1,z_2) = \frac{2|z_1-z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

Ex $\rho(z_1,\infty) = \sqrt{\xi_1^2+\eta_1^2+(\zeta_1-1)^2} = \sqrt{2(1-\zeta_1)}$

$$= \frac{2}{\sqrt{1+r_1^2}} = \frac{2}{\sqrt{1+|z_1|^2}}$$

Metric $\quad \rho(z_1,z_2) = \rho(z_2,z_1)$

Metric

$$\rho(z_1, z_2) = \rho(z_2, z_1)$$

$$\rho(z_1, z_2) \le \rho(z_1, z_3) + \rho(z_3, z_2)$$

$$\rho(z_1, z_2) \ge 0$$

$$\rho(z_1, z_2) = 0 \text{ iff } z_1 = z_2$$

$$\rho(z_1, z_2) \le 2|z_2 - z_1| \le (1+M^2)\,\rho(z_1, z_2)$$

$$|z| < M$$

## Curves

Simple Jordan arc – set of points given by

$z(t) = x(t) + i\,y(t), \quad 0 < t < 1$

$x(t), y(t)$ continuous, real valued f^ns

$\ni \; t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$

simple smooth arc – Jordan arc $\ni$ $\dot{x}(t), \dot{y}(t)$ are continuous

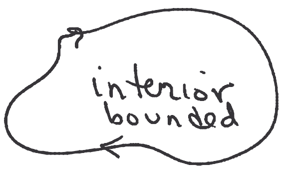
& $\dot{x}^2 + \dot{y}^2 \neq 0$

$\Rightarrow$ continuously turning tangen

simple closed Jordan curve – Jordan simple arc and

$z(t_1) = z(t_2)$ iff $t_1 = 0, t_2 = 1$ or $t_1 = 1, t_2 = 0$

**Jordan Curve Thm** – Every simple closed Jordan curve in $\mathbb{C}$ divides $\mathbb{C}$ into 2 regions (disjoint open sets)




exterior (unbounded)

simple piecewise smooth curve

$t_0 = 0 < t_1 < t_2 < \cdots < t_n < 1 = t_{n+1}$

$\dot{x}, \dot{y}$ are piecewise smooth, $\dot{x}^2 + \dot{y}^2 \neq 0$

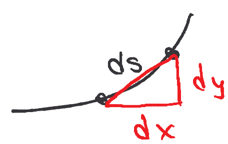
i.e. $\dot{x}, \dot{y}$ are smooth for $t_j < t < t_{j+1}$

$j = 0, \dots, n$

Needed for integration

for example – the length of a curve

$$L = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$



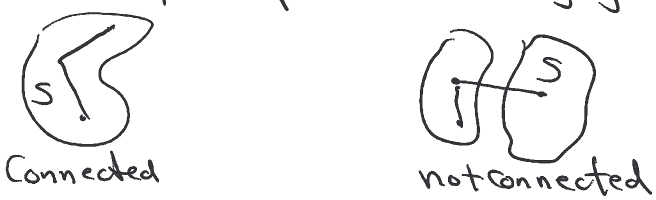

$$L = \int ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

## Regions – S

Connected – if every pair of points can be joined by a simple Jordan arc lying in S

by a simple Jordan arc lying in S




Domain - nonempty, open connected set

Region - may include part or all of the boundary

D-Simply connected - if closed Jordan curves have their interior in D

Functions.

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Let $z \in S$

$$z \longmapsto w = f(z) \quad \Rightarrow \text{pairs } (z, w)$$

$z \in S \subset z$-plane

$w = f(z) \subset w$-plane

$S$ is mapped to the $w$-plane

$z$ is mapped to $w$

$f$ is a mapping

Set of $f(z)$ = Range $= R$

Let $f : S \to R$ - onto

↳ i.e., for every $w \in R$ $\exists$ a $z \in S$ $\ni$ $w = f(z)$.

One to One (1-1) [injection]

If $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$

1-1 and Onto is a bijection $\Rightarrow$ $\exists$ an inverse function

$$z = g(w) \ \ni \ g[f(z)] = z$$

$g$: $R$ onto $S$

Ex $\quad w = z^2$ or $f(z) = z^2$, $\{z \mid |z| \leq 1, 0 \leq \arg z < \pi\}$

Inverse $\quad z = \sqrt{w} \qquad w = re^{i\theta}$

$g(w) = \sqrt{w} \qquad \sqrt{w} = \pm\sqrt{r}\, e^{i\theta/2}$

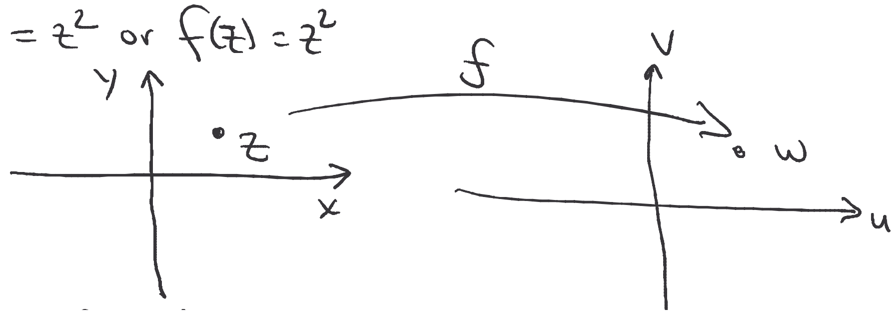
$f(z) = z^2$

$$g[f(z)] = \sqrt{z^2} \overset{?}{=} z$$

Ex $\quad f(z) = \bar{z}$

Ex $\quad f(z) = \dfrac{az+b}{cz+d}$ linear fractional transformation

Ex $\quad w = z^2$ or $f(z) = z^2$



z-plane

$z = x + iy$

w-plane

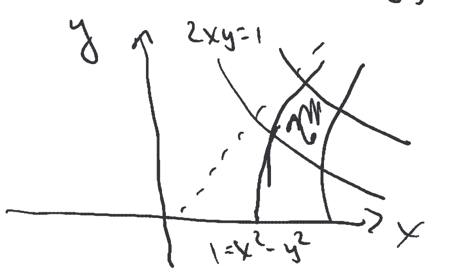
$w = u + iv$

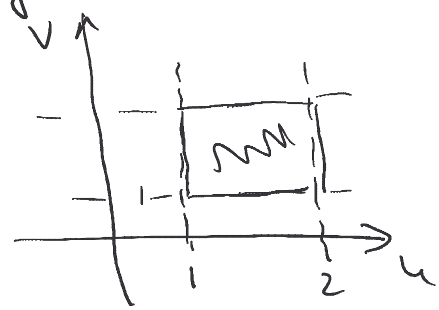
$f(z) = u + iv$

$w = f(z) = u(x,y) + iv(x,y)$

$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$u = x^2 - y^2, \quad v = 2xy$



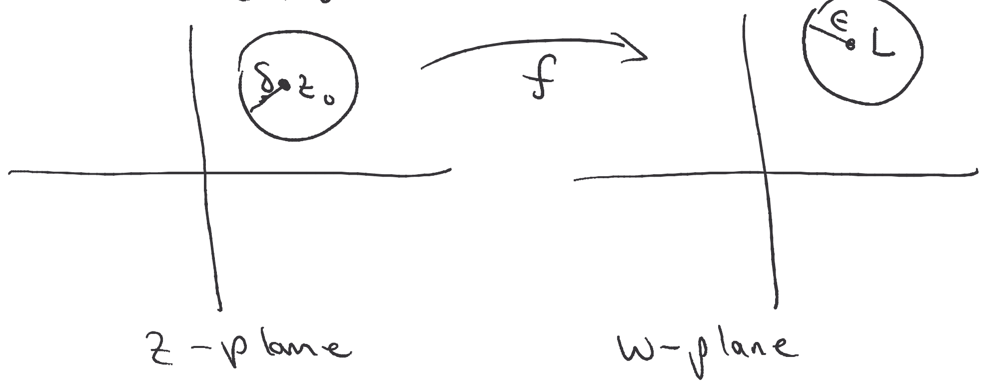
goal - differentiability - we need limits

Let $w = f(z), z \in S$ and let $z_0$ be a limit pt of $S$

If $\exists L \ni. \forall \epsilon > 0 \; \exists \delta \ni. |f(z) - L| < \epsilon, \forall z \in S$

satisfying $0 < |z - z_0| < \delta$, Then

$\lim_{z \to z_0} f(z) = L$.




① f not necessarily defined at $z_0$

② not necessarily defined in $0 < |z - z_0| < \alpha$

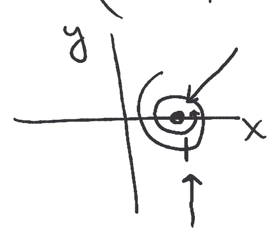
③ May extend our definition to include $z_0 = \infty$

i.e., $\lim_{z \to \infty} f(z)$ or $L = \infty$ i.e. $\lim_{z \to z_0} f(z) = \infty$

But need to use chordal metric: $\rho(f(z), L) < \epsilon$

Ex $f(z) = z^2$ in $S = \{z \mid |z| \leq 1\}$

$\lim_{z \to 1} f(z) = 1$




Pf/ Pick $\epsilon > 0$

Let $|z^2 - 1| < \epsilon$ when $0 < |z - 1| < \delta$ and $z \in S$

$|z-1||\underline{z+1}| < \epsilon$ Note $|z+1| < 2$

Since $|z+1| \leq |z| + |1| < 2$

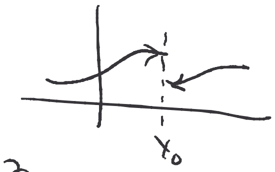
$|z^2 - 1| = |z-1||z+1| < 2|z-1|$

$< 2\delta$

Choose $\delta = \epsilon/2 \Rightarrow |z^2 - 1| < \epsilon$ QED.

Continuity for $x \in \mathbb{R}$, $\lim f(x) = f(x_0)$

__Continuity__ for $x \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = f(x_0)$




Let $w = f(z)$ with domain $S$. $f$ is __continuous at $z_0$__

if $f(z_0) \neq \infty$ and $\forall \epsilon > 0 \ \exists \delta \ni |f(z) - f(z_0)| < \epsilon$

$\forall z \in S$, with $|z - z_0| < \delta$

__$f$ is continuous in $S$__ if it is continuous $\forall z \in S$.

__Ex__ $f(z) = z^2$ Continuous everywhere in finite plane

__Ex__ $f(z) = \frac{1}{z}$ continuous in extended $\mathbb{C}$ w/ $z \neq 0$

text - proves for $z_0 = \infty$ - need chordal metric

__Thm__ $w = f(z)$ is continuous in closed $S$

then $f(z)$ is bounded.

[Bounded - $\exists M > 0 \ni |f(z)| < M, \forall z \in S$]

Note - in general $\delta = \delta(\epsilon, z_0)$

If $\delta$ doesn't depend on $z_0$, then $f$ is uniformly continuous.

__Thm__ $f$ is continuous in closed $S$, then $f$ is uniformly cont.

$

Derivatives



Let $w = f(z)$ in $N_\epsilon(z_0)$ with $f(z_0) \neq \infty$.

Then
$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$
is the derivative of $f(z)$ at $z_0$ if the limit exists and is not $\infty$. Or, we say $f$ is differentiable at $z_0$.

Book - does examples - $z^2$ and $\bar{z}$, $|z|^2$ . . .

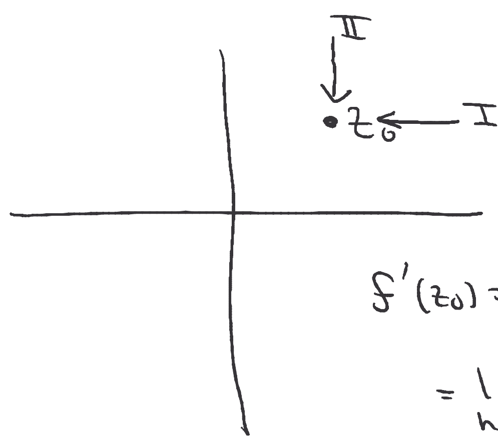
Usual formulae - ✓

$$\frac{(z_0+h)^2 - z_0^2}{h} = \frac{2hz_0 + h^2}{h} \longrightarrow 2z_0$$

## Cauchy-Riemann Equations

$f(z) = u(x,y) + iv(x,y)$

Should get same $f'(z_0)$ no mater what path is taken.




I. $y = y_0 = \text{const}$
$x = x_0 + h$ or $h = x - x_0$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$
$$= \lim_{h \to 0} \frac{u(x_0+h, y_0) + iv(x_0+h, y_0) - u(x_0, y_0) - iv(}{h}$$
$$= \lim_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i\,\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h}$$
$$f'(z_0) = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}$$

II. Let $x = x_0 = \text{const}$
$y = y_0 + h$

$(z = x_0 + i(y_0+h)$
$= z_0 + ih)$

Note $\Delta z = ih$

$$f'(z_0) = \left.\frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right|_{(x_0,y_0)}$$

So we have:
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Need
$$\left[\frac{\partial u}{\partial} - \frac{\partial v}{}\right.$$

So we have:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$\Rightarrow$ Need $\left[\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{array}\right]$ $\underline{\underline{CR}}$

Differentiable
Holomorphic
Analytic

Entire - diff for all $z \in \mathbb{C}$

$\underline{Ex}$ $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$$\left.\begin{array}{l} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{array}\right\} \quad \begin{array}{ll} u_x = 2x & u_y = -2y \\ v_x = 2y & v_y = 2x \end{array}$$

So $u_x = v_y$ & $v_x = -u_y$ ✓

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2yi = 2(x+iy) = 2z$$

$\underline{Thm}$ Sufficient Conditions -

Let $f(z) = u + iv$ in $N_\epsilon(z_0)$ and $u_x, u_y, v_x, v_y$ continuous in $N_\epsilon(z_0)$ $(x_0, y_0)$
and $u, v$ satisfy the CR equations

Then $f$ is differentiable at $z_0$.

$\underline{Ex}$ $f(z) = \bar{z} = x - iy$

$$\left.\begin{array}{l} u(x,y) = x \\ v(x,y) = -y \end{array}\right\} \quad \begin{array}{l} u_x = 1 \\ v_y = -1 \end{array} \quad \text{so } u_x \neq v_y$$

$\underline{Ex}$ $f(z) = \underbrace{\ln\sqrt{x^2+y^2}}_{u(x,y)} + i\underbrace{\tan^{-1}(y/x)}_{v(x,y)}$, $-\frac{\pi}{2} < \tan^{-1}(y/x) < \pi/2$ or $\text{Re}(z) > 0$

$\underline{\text{Test CR}}$

$$\frac{\partial u}{\partial x} = \frac{2x}{2\left(\sqrt{x^2+y^2}\right)^2} \stackrel{?}{=} \frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2} \qquad\qquad = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -u_y$$

$$f'(z) = u_x + iv_x$$

$$f'(z) = u_x + i v_x$$
$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} = \frac{x - iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2}$$
$$= \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$$

<u>Note:</u> $f(z) = \ln z = \ln\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

if $z = re^{i\theta}$, then $\ln z = \ln r + i\theta$

(at some point we will see $\ln z = \ln r + i(\theta + 2n\pi)$
$n \in$ integer

<u>Harmonic functions</u> $u(x,y)$, real valued

$u$ is harmonic iff $\nabla^2 u = 0$

or $u_{xx} + u_{yy} = 0$

$\nabla^2$ - Laplacian, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Given $\nabla^2 u = 0$, let $v$ be the <u>harmonic conjugate</u> to $u$; i.e. $u$ and $v$ satisfy the CR eqns.

Let $f = u + iv$ be diff, with $u_x, u_y, v_x, v_y$ cont.

<u>CR</u> conditions hold and

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -u_{yy}$$

$\Rightarrow \nabla^2 u = 0$ ✓ [Similarly, $\nabla^2 v = 0$]

<u>Ex</u> Consider $u(x,y) = e^x \cos y$. a) Show $\nabla^2 u = 0$
b) Find harmonic conjugate function, $v$.

a)
$$\left.\begin{aligned} u_x &= e^x\cos y \\ u_{xx} &= e^x \cos y \\ u_y &= -e^x \sin y \\ u_{yy} &= -e^x \cos y \end{aligned}\right\} \quad u_{xx} + u_{yy} = 0 \checkmark$$

b) Find $v(x,y) \ni f(z) = u + iv$ is diff.

i) $u_x = v_y \Rightarrow$

$$v_y = e^x \cos y$$

$$\Rightarrow v = \int e^x \cos y \, dy = \boxed{e^x \sin y + C(x)}$$

arbitrary fn

ii) $u_y = -v_x$

$$-e^x \sin y = -\left[e^x \sin y + C'(x)\right]$$

Need $C'(x) = 0$ or $C = \text{const.}$

$$v(x,y) = e^x \sin y + C$$

c) $f(z) = u + iv$

$$= e^x \cos y + i e^x \sin y$$

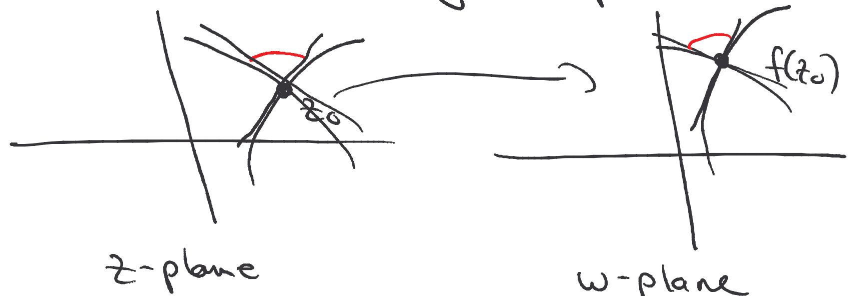
$$= e^x(\cos y + i \sin y) = e^x e^{iy} = e^z$$

d) $f'(z) = u_x + i v_x$

$$= e^x(\cos y + i \sin y) = e^z$$

LFT's are special maps: $f(z)=\dfrac{az+b}{cz+d}$

general properties of analytic maps:




Conformal maps - preserve angles.

Let $w=f(z)$, $z=z_0+\Delta z$

$$\Delta w = f'(z_0)\Delta z + \eta(z_0,\Delta z)\Delta z$$

where $\lim\limits_{\Delta z\to 0}\eta(z_0,\Delta z)=0$

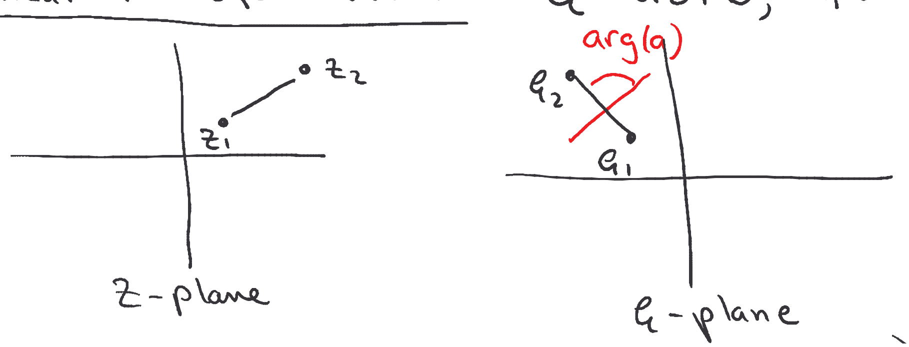
Note
$$f'(z_0)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{f'(z_0)\Delta z+\eta\Delta z}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\left(f'(z_0)+\eta(z_0,\Delta z)\right)$$

$\phi$
$$\arg(\Delta w)=\arg\left(\Delta z\left(f'(z_0)+\eta(z_0,\Delta z)\right)\right)$$
$$=\underset{\theta}{\arg(\Delta z)}+\arg\left(f'(z_0)+\eta(z_0,\Delta t)\right)$$

Let $\Delta z\to 0$.

$$\phi=\theta+\arg(f'(z_0))$$

Linear Transformations    $q=az+b$, $a,b\in\mathbb{C}$



z-plane

ζ-plane

$$\arg(\zeta_2-\zeta_1) = \arg\left(az_2+b-(az_1+b)\right)$$
$$= \arg\left(a(z_2-z_1)\right)$$
$$= \arg a + \arg(z_2-z_1)$$
$$|\zeta_2-\zeta_1| = |a(z_2-z_1)|$$
$$= |a|\,|z_2-z_1|$$

$\leftarrow$ stretching/shrinking

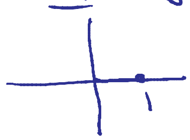
$\Rightarrow$ Polygons, linesegments are rotated by arg(a) and multiplied by $|a|$. $\Rightarrow$ images are similar to original domains.

Similarly, circles $\to$ circles, lines $\to$ lines

<u>Ex</u> $f(z) = z+\beta$

$a=1$, $\arg(1) = \arg(re^{i\theta}) = 0$

no rotation



$|a| = |1| = 1$

$\Rightarrow$ no effect on rotation, scaling.

<u>Inversion</u> $\zeta = \frac{1}{z}$, nearly conformal

$z \neq 0$

Circles $\to$ Circles

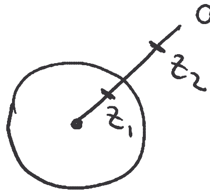
Eqn of a circle $|z-z_0| = \rho$ or $z-z_0 = \rho e^{i\theta}$

$$\left|z - \frac{p}{\alpha}\right|^2 = \frac{|p|^2}{\alpha^2} - \frac{\beta}{\alpha}, \quad \alpha\beta<|p|^2$$

$$\Rightarrow \quad \alpha|z|^2 - \bar{p}z - p\bar{z} + \beta = 0$$

$\zeta = 1/z \quad \Rightarrow \quad \beta|\zeta|^2 - p\zeta - \bar{p}\bar{\zeta} + \alpha = 0$

If $\alpha = 0$, $\quad -\bar{p}z - p\bar{z} + \beta = 0$ is a line and this maps to a circle.

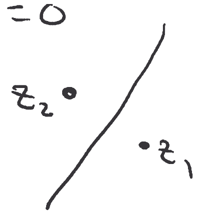


product of distance from center = radius squared

For a line: $\alpha = 0$

$$-\bar{p}z_1 - p\bar{z}_2 + \beta = 0$$

For a line: $\alpha = 0$




$$-\bar{p} z_1 - p \bar{z}_2 + \beta = 0$$

Bilinear / Möbius / Linear Fractional Transformations

$$\zeta = \frac{az+b}{cz+d} = f(z)$$

$$\zeta = \frac{a\left(z + \frac{d}{c} - \frac{d}{c}\right) + b}{c(z + d/c)} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

Composition: $\zeta_1 = cz + d$, linear

$\zeta_2 = \frac{1}{\zeta_1}$, inversion

$\zeta = \frac{a}{c} + \underbrace{\left(b - \frac{ad}{c}\right)}_{\neq 0,\ \text{else}\ \mathbb{C} \to \text{pt.}} \zeta_2$, linear

i.e. $ad - bc \neq 0$

1–1 conformal mapping of the extended $z$-plane to the extended $\zeta$-plane.

generalized circles $\longrightarrow$ generalized circles

<u>Composition of 2 bilinear transformations:</u>

$$f(z) = \frac{az+b}{cz+d}, \quad g(z) = \frac{a'z+b'}{c'z+d'}$$

Let $f(z)$ be represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then the coeff's for $f(g(z))$ are $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Inverse Transformation to $f(z)$: $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Circles –

How many points determines a circle?

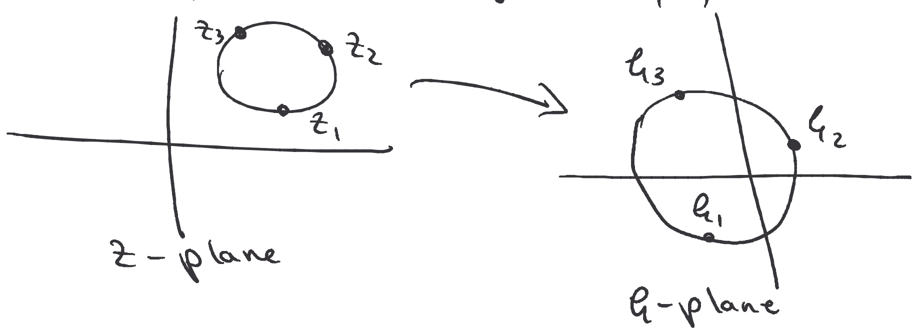
$$ax^2 + ay^2 + bx + cy + d = 0$$

$$x^2 + y^2 + \frac{b}{a}x + \frac{c}{a}y + \frac{d}{a} = 0$$

Need 3 points. $\Rightarrow$ $b/a$, $c/a$, $d/a$.

Let $\zeta_1, \zeta_2, \zeta_3$ be the images of $z_1, z_2, z_3$

Let $w_1, w_2, w_3$ be the images of $z_1, z_2, z_3$




$w_1 = \dfrac{az_1+b}{cz_1+d}$, etc $\Rightarrow$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_2)(w_1-w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_2)(z_1-z_3)}$$

<u>Ex</u> Find bilinear transformation mapping $|z|=1$ into $\operatorname{Im} w = 0$.


Choose $\underset{z_i}{1, i, -1}$ and $\underset{w_i}{1, 0, -1}$

$$\Rightarrow \quad \frac{(w-1)(0-(-1))}{(w-0)(1-(-1))} = \frac{(z-1)(i+1)}{(z-i)(2)}$$

Solve: $\quad w = \dfrac{z-1}{1-iz}$ (?)

Homework – pg 26

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HW #2 - Length of Paths in $\mathbb{C}$

2a) $z = a\cos 2\pi t + ib\sin 2\pi t$

$x = a\cos 2\pi t$, $y = b\sin 2\pi t$

$\frac{x}{a} = \cos 2\pi t$, $\frac{y}{b} = \sin 2\pi t \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$L = \int_0^1 \sqrt{a^2\sin^2 2\pi t + b^2\cos^2 2\pi t}\, 2\pi\, dt$, Let $\theta = 2\pi t$

$= b\,E(2\pi, e)$, $b > a$ – Elliptic function

$\longrightarrow \int_0^\phi \sqrt{1 - k^2\sin^2\theta}\, d\theta \equiv E(\phi, k)$

Elliptic Function
(Incomplete)

$e = \frac{\sqrt{b^2 - a^2}}{b}$

(2b) $|z| = 1 - \cos(\arg z)$ $\quad 0 \le \arg z \le 2\pi$

Polar form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$|z| = r$, $\arg z = \theta \Rightarrow r = 1 - \cos\theta$ cardiod

$x = r\cos\theta = (1 - \cos\theta)\cos\theta$
$y = r\sin\theta = (1 - \cos\theta)\sin\theta$

Note: you will need $\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$

(2c) $z = te^{it}$

$x = t\cos t$, $y = t\sin t$

$\Rightarrow L = \int_0^{2\pi} \sqrt{1 + t^2}\, dt$, $t = \tan\theta$

(2d) $x = t - \sin t$
$y = 1 - \cos t$ $\quad 0 \le t \le 2\pi$

Transcendental Functions

Thursday, February 08, 2007
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Algebraic Functions — $az+b$, polynomial, rational, $z^2$

Transcendental Functions – $\sin z$, $\cosh z$, $e^z$, $\ln z$, ...

① $e^z$: $\begin{cases} f(z) \text{ -analytic} \\ f'(z) = f(z) \\ f(x) = e^x \end{cases}$  $f = u+iv$

$u_x + iv_x = u + iv$, etc.

$\Rightarrow u_x = u$ or $u = e^x g(y)$, $v = e^x h(y)$

But $u_x = v_y$

$v_x = -u_y$

$\Rightarrow \begin{cases} e^x g = e^x h' \\ e^x h = -e^x g' \end{cases}$ or $\begin{cases} g = h' \\ h = -g' = -h'' \end{cases}$

or $h'' + h = 0$

$h(y) = a\cos y + b\sin y$

$g(y) = -a\sin y + b\cos y$

$f(x) = e^x \Rightarrow g(0) = 1,\ h(0) = 0 \Rightarrow a = 0, b = 1$

$f(z) = e^x(\cos y + i\sin y) = e^{x+iy} = e^z$

Properties — $|e^z| = |e^x e^{iy}| = |e^x| = e^x$

$\overline{e^z} = e^{\bar{z}}$

$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\}$  $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\begin{aligned} \sin z &= \sin(x+iy) \\ &= \sin x\cos(iy) + \sin(iy)\cos x \\ &= \sin x\cosh y + i\sinh y\cos x \end{aligned}$$

since $\sin(iy) = \dfrac{e^{-y}-e^{y}}{2i} = \dfrac{-1}{i}\sinh y$

$\cosh z = \dfrac{e^z + e^{-z}}{2}$

$\sinh z = \dfrac{e^z - e^{-z}}{2}$

Logarithm  $w = \ln z \iff z = e^w$

$re^{i\theta} = e^w$

$2\pi ki$ $\ln r + i\theta$

$$re^{i\theta} = e^{w}$$

$$e^{2\pi K i} e^{\ln r + i\theta} = e^{w} \quad K\text{-integer}$$

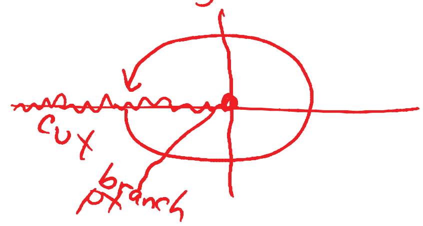
$$w = \ln r + i(\theta + 2\pi K)$$

$$= \ln\sqrt{x^2+y^2} + i\left(\tan^{-1}(y/x) + 2\pi K\right)$$

multivalued

Each $K \Rightarrow$ different branch

$-\pi \le \theta \le \pi$, $K=0$ Principal Branch




$$\text{Log } z = \ln\sqrt{x^2+y^2} + i\tan^{-1} y/x$$

Inverse functions $\cos^{-1} z$, $\sinh^{-1} z$, etc.

$$w = \sin^{-1} z \iff z = \sin w$$

$$= \frac{e^{iw} - e^{-iw}}{2i}$$

$$2iz = e^{iw} - \frac{1}{e^{iw}}$$

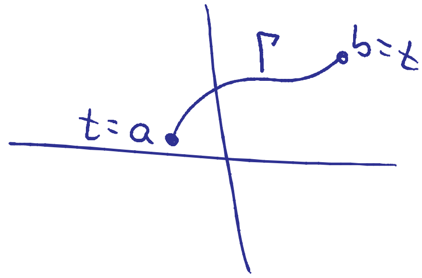
$$2ize^{iw} = (e^{iw})^2 - 1$$

or $(e^{iw})^2 - 2ize^{iw} - 1 = 0$

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2+4}}{2} = iz \pm \sqrt{1-z^2}$$

$$iw = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$\int_\Gamma f(z)dz$ path integral

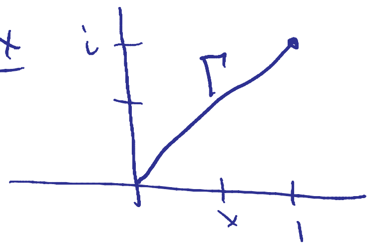



$\Gamma = \{(x(t), y(t)) \mid a \le t \le b\}$

$z = x(t) + iy(t)$

$$\int_\Gamma f(z)dz = \int_a^b f(x(t)+iy(t))[x' + iy']dt$$

Ex




$I = \int_\Gamma z^2 dz$

$z = x + ix, \quad x \in [0,1]$

$dz = (1+i)dx$

$$I = \int_0^1 (1+i)^2 x^2 (1+i)dx$$

$$= (1+i)^3\left(\frac{1}{3}\right) = \frac{(i-1)}{3}2$$

$(1+i)^3 = 1 + 3i + 3i^2 + i^3 = -2 + 2i$

Dettman - Ch 3

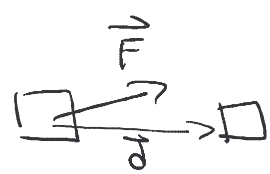
Starts with path integrals $\int_C P\,dx + Q\,dy$

where $P, Q$ are real valued functions of two variables

$C$ - path in $\mathbb{R}^2$

Application - Work $\quad W = F \cdot \text{dist}$

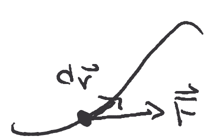
$W = \vec{F} \cdot \vec{d}$




$\vec{F} = P(x,y)\hat{i} + Q(x,y)\hat{j}$

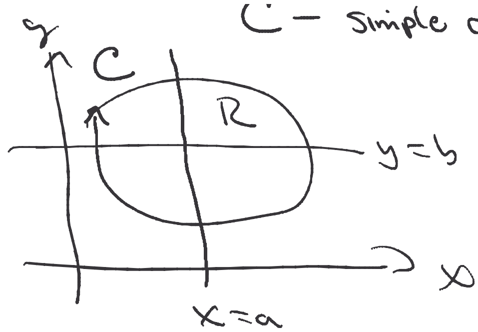
$d\vec{r} = dx\hat{i} + dy\hat{j}$

$W = \int_C \vec{F} \cdot d\vec{r}$




Green's Lemma

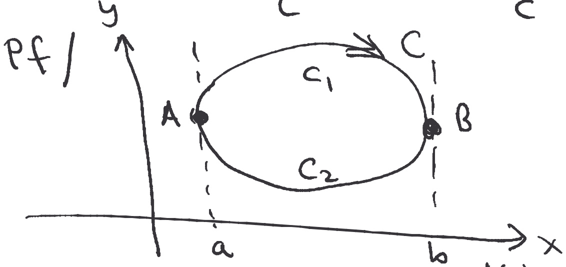

$C$ - simple closed contour and $x=a$, $y=b$ intersect $C$ in at most 2




C – simple closed contour and $x=a$, $y=b$ intersect C in at most 2 points. $P(x,y)$, $Q(x,y)$ are real valued, continuous and have continuous first partial derivatives in $\underbrace{C \cup \text{interior of } C}_{R}$

Then $$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \iint_R (Q_x - P_y)\,dx\,dy$$

Pf/




Consider $$-\iint_R \frac{\partial P}{\partial y}\,dy\,dx \equiv I$$

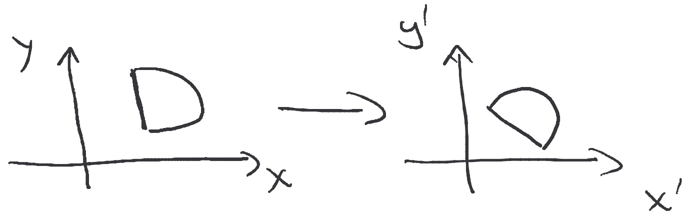
Then $$I = -\int_a^b \left(\int_{g(x)}^{h(x)} \frac{\partial P}{\partial y} dy\right) dx = -\int_a^b [P(x,h) - P(x,g)]\,dx$$

$$= \int_C P(x,y)\,dx$$ being careful of path direction.

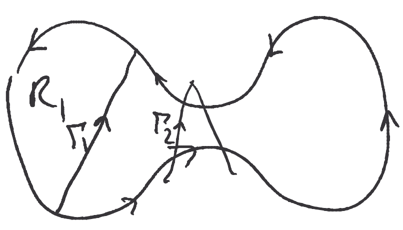
Similarly $$\iint_R \frac{\partial Q}{\partial x}\,dy\,dx = \int_C Q\,dy$$

Variations

① Rotate axes




② More Complex Regions




$$\int_C P\,dx + Q\,dy \overset{?}{=} \iint_R (Q_x - P_y)\,dx\,dy$$

But $$\int_{C_1} P\,dx + Q\,dy = \iint_{R_1} (Q_x - P_y)\,dx\,dy$$

$$\Rightarrow \sum_{i=1}^{n} \int_{C_i} P\,dx + Q\,dy = \sum_{i=1}^{n} \iint_{R_i} (Q_x - P_y)\,dx\,dy = \iint_{R} (\quad)\,dx\,dy$$

Schematically,

$$\int_{C_1} = \int_{C_1'} + \int_{\Gamma_1}$$

$$\int_{C_2} = \int_{C_2'} + \int_{\Gamma_2} - \int_{\Gamma_1}$$

$$\vdots$$

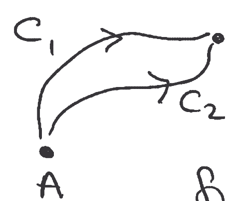
$$\int_{C_n} = \int_{C^{n\,\prime}} - \int_{\Gamma_{n-1}}$$

Note: the path integral from a to b is negated as the direction is reversed.

$$\text{Sum} \quad \sum_{i=1}^{n} \int_{C_i} = \int_{C}$$

<u>Path Independence</u>

$\int_C P\,dx + Q\,dy$ is the same for any path between two given pts.




Path Indep $\Rightarrow$ $\oint_C P\,dx + Q\,dy = 0$ $\forall$ simple closed C

$$\oint_C = \oint_{C_1} - \oint_{C_2} = 0$$

## <u>Contours in z-plane</u>

Let $f(z) = u + iv$ in a simply connected domain D

$u, v$ have continuous 1st derivatives and $u_x = v_y$, $u_y = -v_x$.

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -u_y\,dx + u_x\,dy \quad \Rightarrow \quad v = \int dv = \int -u_y\,dx + u_x\,dy$$

Note if $\nabla^2 u = 0$ then one can get v from u for v the harmonic conjugate

<u>3.2</u> $\int_a^b f(z)\,dz$ If $f(z)$ continuous on a smooth arc C,

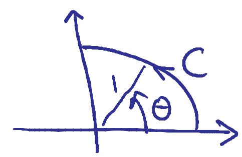
3.2 $\int_a f(z)\,dz$ If $f(z)$ continuous on a smooth arc $C$, The $\int_C f dz$ exists.

Ex $\displaystyle\int_a^b f(z)dz = \int_{t_a}^{t_b} (u+iv)(\dot{x}+i\dot{y})dt$ $\quad z(t)=x(t)+y(t)i$

$dz = (\dot{x}+i\dot{y})dt$

$$= \int_{t_a}^{t_b} [(u\dot{x}-v\dot{y})+i(v\dot{x}+u\dot{y})]dt$$

Ex $\displaystyle\int_C \bar{z}\,dz$




arc of unit circle

method I $\quad x=\cos\theta$

$y=\sin\theta$

$\bar{z}=\cos\theta - i\sin\theta$, $z=\cos\theta + i\sin\theta$

$dz = (-\sin\theta + i\cos\theta)d\theta$ — finish ...

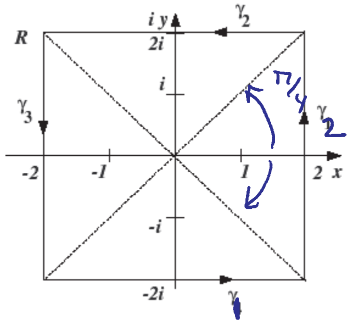
method II $\quad$ On unit circle, $z=e^{i\theta}$

$\Rightarrow \bar{z}=e^{-i\theta}$, $dz = ie^{i\theta}d\theta$

$$\int_C \bar{z}\,dz = \int_0^{\pi/2} (e^{-i\theta})\, i e^{i\theta} d\theta = i\int_0^{\pi/2} d\theta$$

So far – $\int_C f(z)\,dz$ for $z=z(t)$ along $C$

$dz/z$




$\gamma_1$: $z = 2+iy \quad y\in[-2,2]$

$dz = i\,dy$

$$\int_{\gamma_1}\frac{dz}{z} = \int_{-2}^{2}\frac{i\,dy}{2+iy} = \ln(2+iy)\Big|_{-2}^{2}$$

$$= \ln(2+2i) - \ln(2-2i)$$

$$= \ln\left(2\sqrt{2}\,e^{i\pi/4}\right) - \ln\left(2\sqrt{2}\,e^{-i\pi/4}\right)$$

$$= i\pi/2 \quad \checkmark$$

$$\begin{aligned}
\oint_{\gamma_1}\frac{dz}{z} &= \int_{-2}^{2}\frac{dx}{x-2i} \\
&= \ln|x-2i|_{-2}^{2} \\
&= (\ln(2\sqrt{2})+\frac{7\pi i}{4}) - (\ln(2\sqrt{2})+\frac{5\pi i}{4}) \\
&= \frac{\pi i}{2}.
\end{aligned}$$

Similarly, the integral along the top segment is computed as

$$\begin{aligned}
\oint_{\gamma_3}\frac{dz}{z} &= \int_{2}^{-2}\frac{dx}{x+2i} \\
&= \ln|x+2i|_{2}^{-2} \\
&= (\ln(2\sqrt{2})+\frac{3\pi i}{4}) - (\ln(2\sqrt{2})+\frac{\pi i}{4})
\end{aligned}$$

$$= \frac{\pi i}{2}.$$

The integral over the right side is

$$\begin{aligned}
\oint_{\gamma_2} \frac{dz}{z} &= \int_{-2}^{2} \frac{idy}{2+iy} \\
&= \ln|2+iy|_{-2}^{2} \\
&= (\ln(2\sqrt{2}) + \frac{\pi i}{4}) - (\ln(2\sqrt{2}) - \frac{\pi i}{4}) \\
&= \frac{\pi i}{2}.
\end{aligned}$$

Finally, the integral over the left side is

$$\begin{aligned}
\oint_{\gamma_4} \frac{dz}{z} &= \int_{2}^{-2} \frac{idy}{-2+iy} \\
&= \ln|-2+iy|_{-2}^{2} \\
&= (\ln(2\sqrt{2}) + \frac{5\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{3\pi i}{4}) \\
&= \frac{\pi i}{2}.
\end{aligned}$$

Total $\oint_C \frac{dz}{z} = 2\pi i$

There is an easier way! - Cauchy's Thm

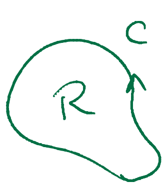
Section 3.3

Thm — Let $f(z)$ be analytic in a simply connected domain $D$. Then $\oint_C f(z)dz = 0$ for $C$ a simple closed contour.

Version I Let $f(z) = u + iv$ and $z = x + iy$. Assume $u, v$ satisfy the CR equations. Then $\oint_C f(z)dz = 0$

Pf/ $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$

$$= \oint_C u\,dx - v\,dy + i\oint_C u\,dy + v\,dx$$



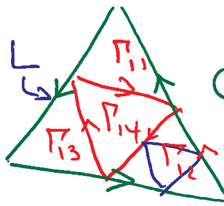

Consider $\oint_C u dx - v dy = \iint_R \underbrace{\cancel{(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})}}_{0 \text{ by } CR} dxdy = 0$

Similarly $\oint_C v dx + u dy = 0.$

Version II Our text - 3 Steps

1. Triangles
2. Closed Polygons
3. General Contours.

Step ① Lemma – Assume $\int_C f(z)dz = I \neq 0$




Connect midpoints on sides

$$\int_C = \int_{\Gamma_{11}} + \int_{\Gamma_{12}} + \int_{\Gamma_{13}} + \int_{\Gamma_{14}}$$

$$\int_C \neq 0 \Rightarrow \int_{\Gamma_{1j}} \neq 0 \text{ for some } j$$

Let $J_1 = \max_j \left| \int_{\Gamma_{1j}} f\,dz \right| \Rightarrow \left| \int_C f\,dz \right| \leq 4 J_1$

Repeat process for triangle $\Gamma_{1j} \equiv \Gamma_1$

$$\Rightarrow \left| \int_C f\,dz \right| \leq 4^n \left| \int_{\Gamma_n} f\,dz \right|$$

Interiors $C^*, \Gamma_1^*, \Gamma_2^*, \ldots$

& $R_0 = C^* \cup C$, $R_1 = \Gamma_1^* \cup \Gamma_1$

So, $R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$ Closed, nested sequence of sets.

Intersection = point $z_0 \in D$ (Thm 1.4.2)

$$\left.\begin{array}{l} d(R) = \text{diam}(R) \\ d(R_0) \leq L/2 \\ d(R_1) \leq L/4 \\ \text{etc.} \end{array}\right\} \quad d(R_n) = \frac{L}{2^{n+1}}$$

Consider $f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta(z,z_0)(z-z_0)$

given $\epsilon > 0 \ \exists \ \delta(\epsilon) \ni$ $|\eta| < \epsilon$ whenever $|z-z_0| < \delta$

$$\left| \int_{\Gamma_n} f(z)dz \right| = \left| \int_{\Gamma_n} [f(z_0) + f'(z_0)(z-z_0) + \eta(z,z_0)(z-z_0)]dz \right|$$

$$= \left| \int_{\Gamma_n} \eta(z,z_0)(z-z_0)\,dz \right|$$

$$\leq \int_{\Gamma_n} |\eta(z,z_0)(z-z_0)|dz$$

$d(R_n) < \delta$

$$= \int_{\Gamma_n} |\eta|\, \underline{|z-z_0|}\, dz \leq \epsilon \frac{L}{2^{n+1}} \frac{L}{2^n} = \epsilon \frac{L^2}{2 \cdot 4^n}$$

Need

3.2.2 $\int_a^b dz = b-a$

3.2.3 $\int_\alpha^\beta z\,dz = \frac{1}{2}(\beta^2 - \alpha^2)$

$$\left\} \begin{array}{l} \int_{\Gamma_n} dz = 0 \\ \int_{\Gamma_n} z\,dz = 0 \end{array}\right.$$

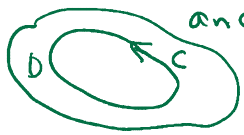
$$|I| = \left| \oint_C f(z)dz \right| \leq 4^n \left| \int_{\Gamma_n} f(z)dz \right| \leq \cancel{4^n} \left( \frac{\epsilon L^2}{2 \cdot \cancel{4^n}} \right) = \frac{\epsilon L^2}{2}$$

$|I| \leq \frac{L^2}{2}\epsilon$ for any $\epsilon > 0$. $\Rightarrow \underline{\underline{|I| = 0}}$ ∅

# Implications

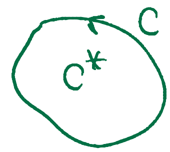
Thursday, February 22, 2007
5:10 PM

3.3.1 $f(z)$ is analytic in a simply connected domain $D$ and $C \subset D$, simple closed contour, Then



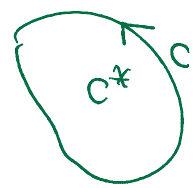

$$\int_C f(z)dz = 0 \qquad (\text{pg } 86)$$

3.4.2 $f(z)$ is analytic within $C$ and continuous in $C \cup C^*$, Then $\int_C f(z)dz = 0$

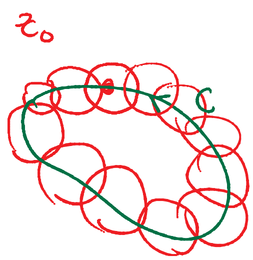



3.4.1 $f(z)$ analytic inside and on $C \Rightarrow$

$$\int_C f(z) dz = 0$$




Pf/




① f anal at $z_0$ $\Rightarrow$ f anal in Disk about $z_0$

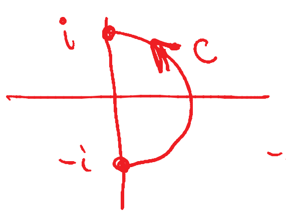
② $\infty$ family of disks covering C

③ Heine-Borel – we can find a finite sub covering

$$D = C^* \cup \Big[\bigcup_\alpha D_\alpha\Big]$$

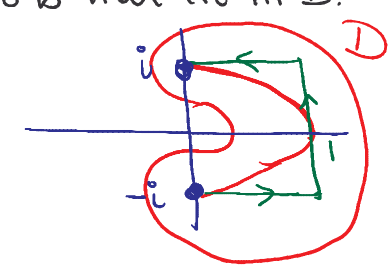
3.4.3 $f(z)$ is analytic in a simply connected domain Then $\int_a^b f(z)dz$ is path (contour) independent for contours from a to b that lie in D.

Ex 3.4.1 pg 93 $\quad \int_{-i}^{i} \frac{dz}{z}$




$z = e^{i\theta}$

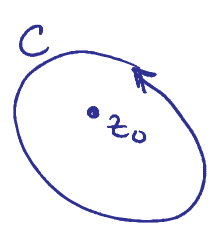
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$




3.4.4 C – simple closed contour

$$\int_C \frac{dz}{z - z_0} = \pm 2\pi i$$

depending on the direction




Uses – HW Ex 3.2.6 (pg 86)

Uses – HW Ex 3.2.6 (pg 86)

Pf $I = \int_C \frac{dz}{z-z_0} = 2\pi i$ for $C: |z-z_0| = \rho$

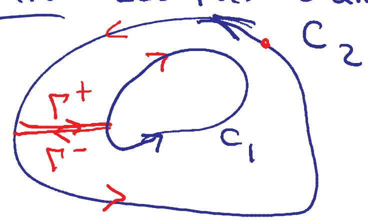
Let $z - z_0 = \rho e^{i\theta}$

& $dz = i\rho e^{i\theta} d\theta$

$$I = \int_0^{2\pi} \frac{i\rho e^{i\theta} d\theta}{\rho e^{i\theta}}$$

$$= i\theta \Big|_0^{2\pi} = 2\pi i$$

3.4.5 Let $f(z)$ be analytic on $C_1, C_2$ and between $C_1$ & $C_2$.

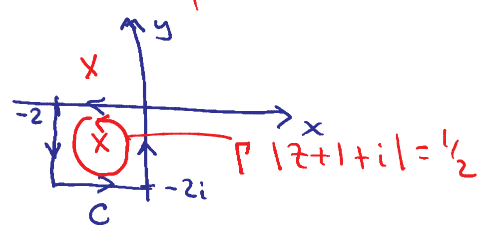



Then $\int_{C_{2+}} f(z)dz = \int_{C_{1+}} f(z)dz$

$$\int_{C_2} + \int_{C_1^-} + \cancel{\int_{\Gamma^+} + \int_{\Gamma^-}}^{\,0} = 0$$

$$\int_{C_2} - \int_{C_1} = 0 \quad \Rightarrow \quad \int_{C_2} = \int_{C_1}$$

Ex 3.4.3 $\int_C \frac{dz}{z^2+2z+2}$




$0 = z^2 + 2z + 2 = (z - z_+)(z - z_-)$

$= z^2 + 2z + 1 + 1$

$0 = (z+1)^2 + 1$

$z + 1 = \pm i$

$z = -1 \pm i$

So, $z^2 + 2z + 2 = (z+1-i)(z+1+i)$

Note $\frac{1}{(z+1-i)(z+1+i)} = \left[\frac{1}{z+1-i} + \frac{-1}{z+1+i}\right]\frac{1}{2i}$

$= \frac{z+1+i-(z+1-i)}{(z+1-i)(z+1+i)} \frac{1}{2i}$

$$\int_C \frac{dz}{z^2+2z+2} = \frac{1}{2i}\int_C \frac{dz}{\underbrace{z+1-i}_{z-(-1+i)}} - \frac{1}{2i}\int_C \frac{dz}{\underbrace{z+1+i}_{z-(-1-i)}}$$

$\downarrow$

vanishes by Cauchy's Thm

$$\int \frac{dz}{z+1+i} = \int \frac{dz}{z+1+i} = \int \frac{dz}{z-(-1-i)} = 2\pi i$$

$$\int_C \frac{dz}{z+1+i} = \int_\Gamma \frac{dz}{z+1+i} = {}^{0}\int_\Gamma \frac{dz}{z-(-1-i)} = 2\pi i$$

So $$\int_C \frac{dz}{z^2+2z+2} = -\frac{1}{2i}(2\pi i) = \boxed{-\pi}$$

<u>Ex</u> <u>3.4.4</u> $$\int_{\substack{C_+ \\ |z|=2}} \frac{dz}{z^2-1}$$